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Coláiste na Tríonóide, Baile Átha Cliath
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The Twistor Origin of Integrability

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School of Mathematics

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Motivation

Motivation

> "Using a term like non-linear science is like referring to the bulk of zoology as the study of non-elephant animals."
> — *Stanislaw Ulam.*

Motivation

- Much of theoretical physics is governed by *non-linear* differential equations, and most resist analytic treatment.

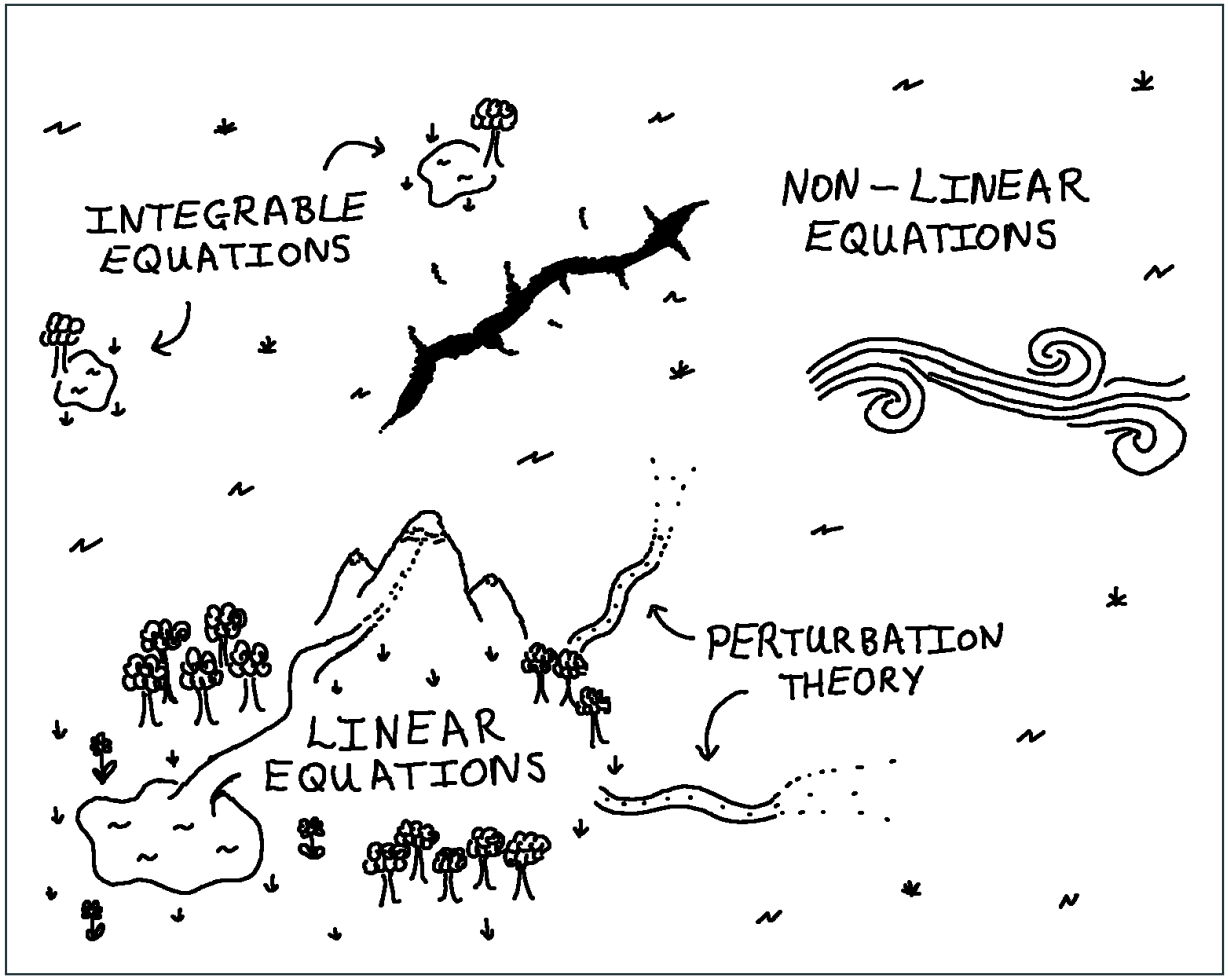
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- Much of theoretical physics is governed by *non-linear* differential equations, and most resist analytic treatment.
- There is a rare subset of these non-linear systems called **integrable systems**.
- Integrable equation are to some degree solvable, i.e. many explicit solutions exist and we can find them.

The Landscape of Differential Equations[1]



[1]Cole 2024.

The Origin of Integrability

Key Question

Why are these particular equations special, and is there a higher-dimensional *master theory* from which they all come from?

The ASDYM Equation

Anti-Self-Dual Yang–Mills as a Unifying Theory

- Yang–Mills theory: a gauge field A with curvature

$$F = dA + A \wedge A.$$

[2] Mason and Woodhouse 1996.

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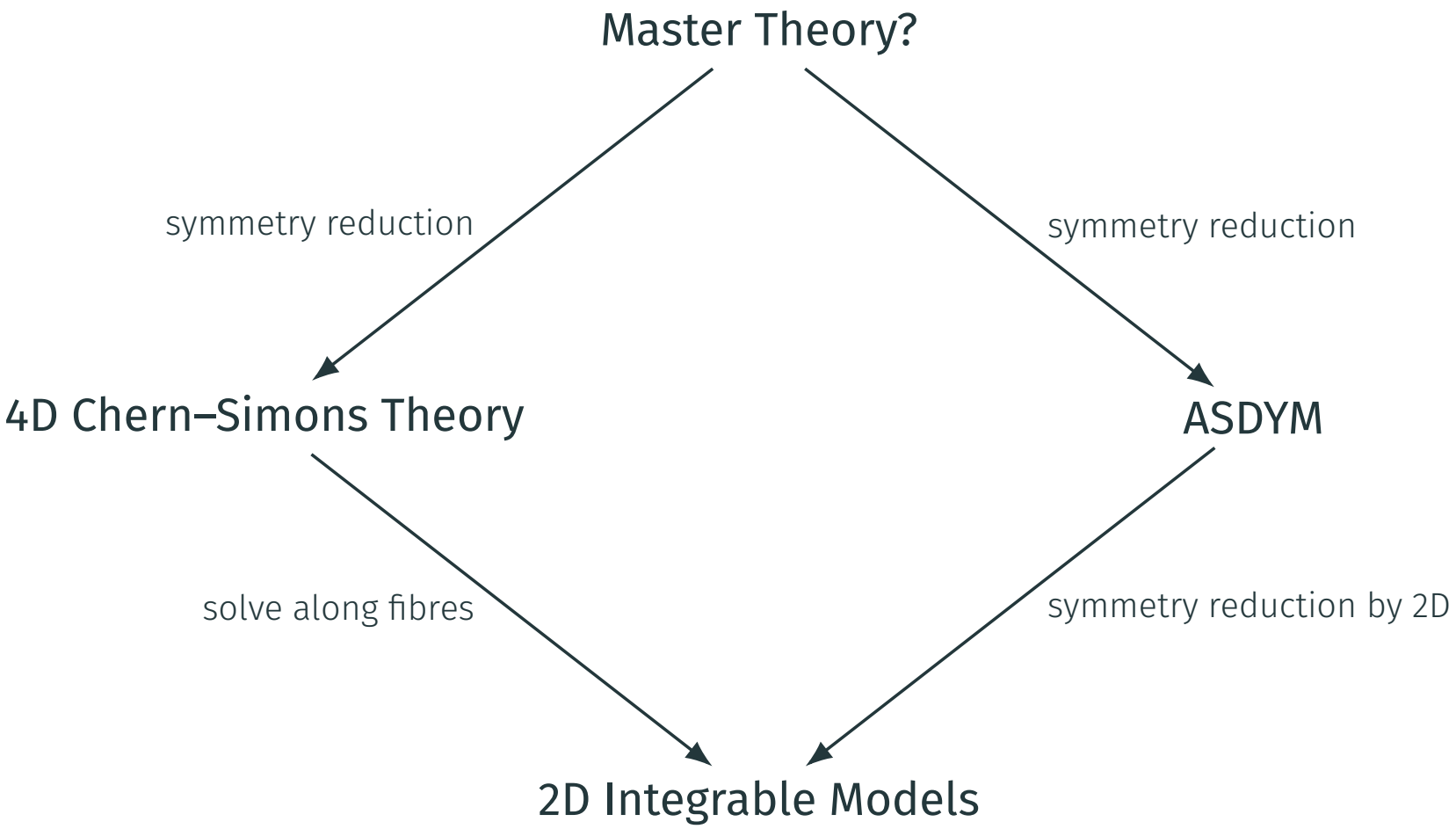
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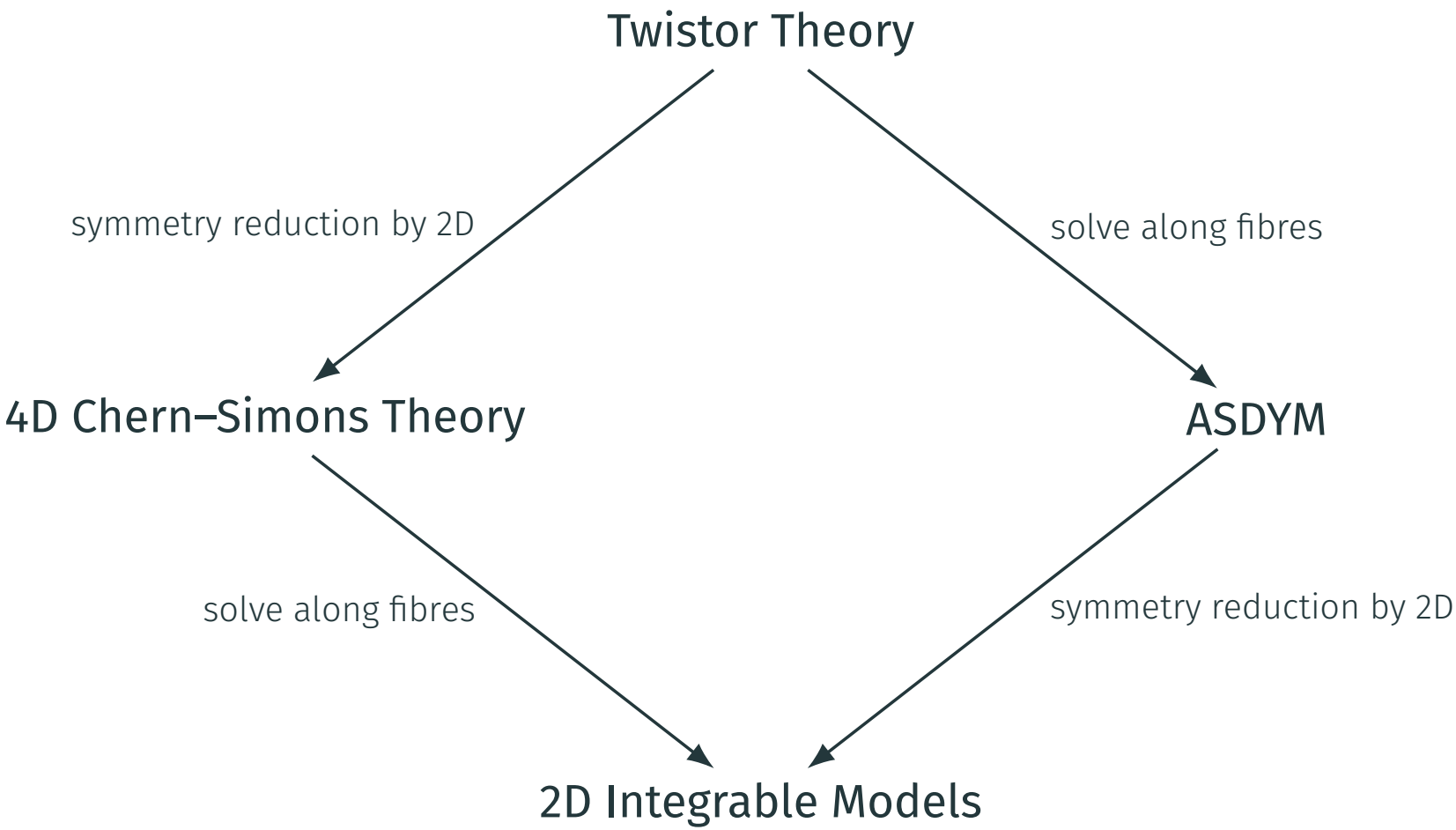
- Many 1+1D integrable models (sine-Gordon, KdV, NLS, PCM, ...) arise as *dimensional reductions* of ASDYM.[2]

[2] Mason and Woodhouse 1996.

The Initial Picture



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Twistor Theory

Complexified Minkowski Space

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- One can go between the vector and spinor indices as follows:

$$v^{\alpha\dot{\alpha}} = \frac{\sigma_a^{\alpha\dot{\alpha}}}{\sqrt{2}} v^a = \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix}.$$

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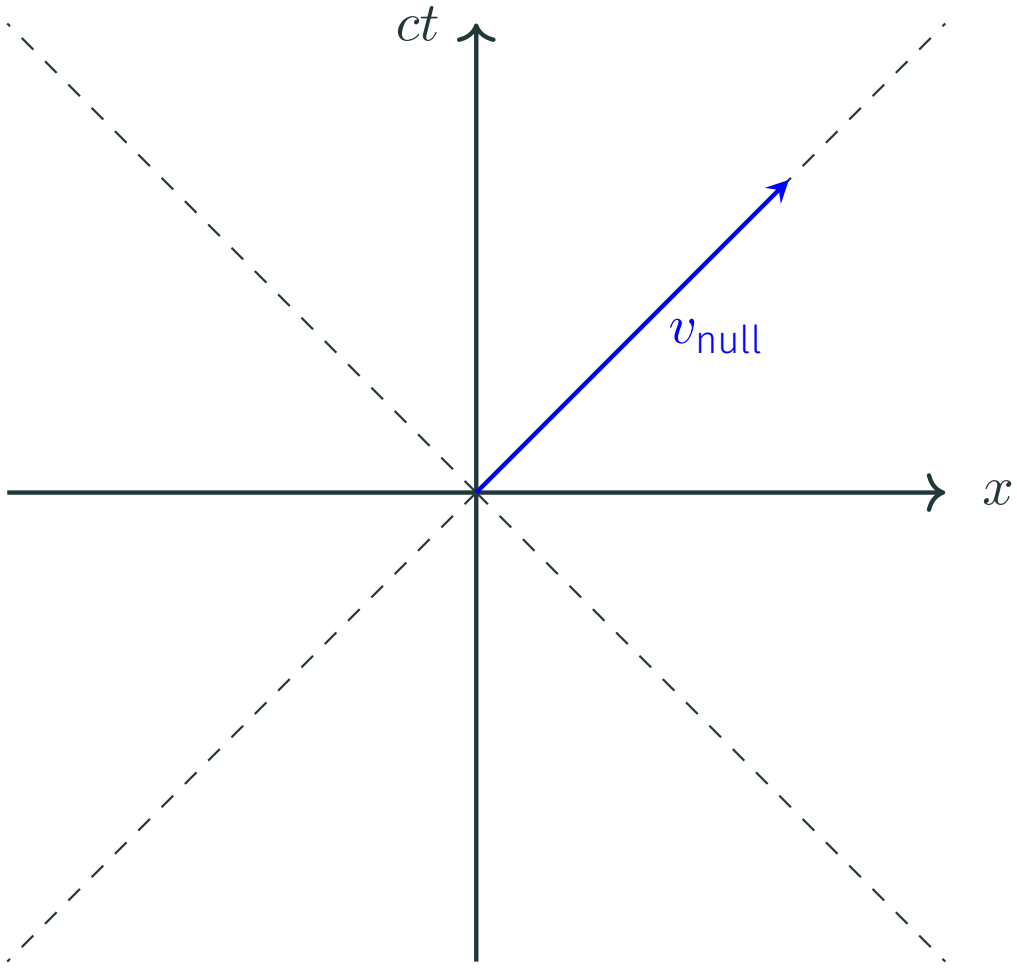
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- Null vectors have length zero, i.e.

$$\eta_{ab} v^a v^b = 2 \det(v^{\alpha\dot{\alpha}}) = \left(v^0\right)^2 - \left(v^1\right)^2 - \left(v^2\right)^2 - \left(v^3\right)^2 = 0$$

Null Vector Example: 1+1D Minkowski Space



Null Vector Factorisation

Key Fact

Null vectors factorise into a pair of spinors, and a pair of spinors multiply to give a null vector,

$$v_{\text{null}}^{\alpha\dot{\alpha}} = \kappa^{\alpha}\,\tilde{\kappa}^{\dot{\alpha}}.$$

Twistor Space

- **Twistor space** $\mathbb{PT}$ is (an open subset of) $\mathbb{CP}^3$, with homogeneous coordinates

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- The tricky part is defining a relation between $\mathbb{PT}$ and space-time.
- This is a non-local relationship and is referred to as the twistor correspondence.

The Twistor Correspondence

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Applying the projective nature to this, this defines a $\mathbb{CP}^1 \subset \mathbb{PT}$, so each point $x \in \mathbb{M}_{\mathbb{C}}$ corresponds to a Riemann sphere in $\mathbb{PT}$.

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- Viewing a point Z in twistor space as the intersection of two lines X and Y, we have

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Finding all of the lines that intersect at a point Z in twistor space, we can say that it corresponds to a null 2-plane in $\mathbb{M}_{\mathbb{C}}$.

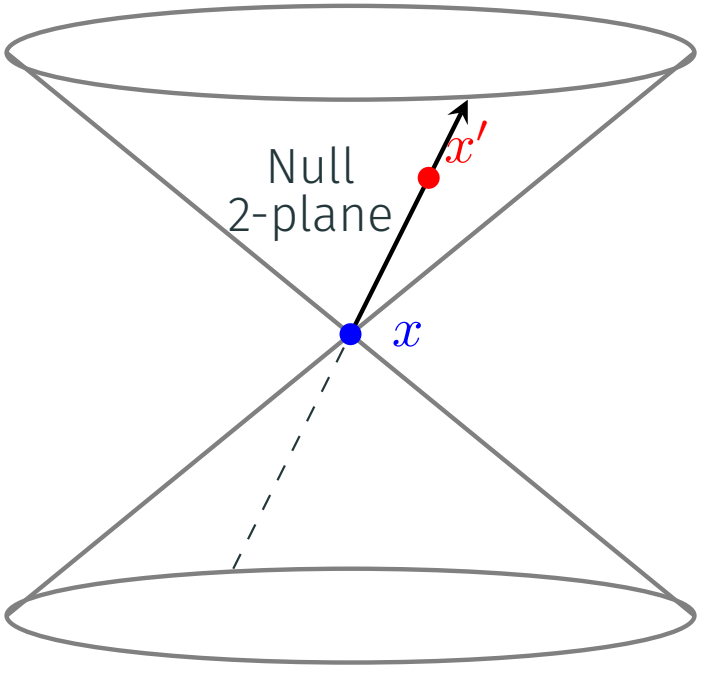
Summary of Twistor Correspondance

$$\text{Points in } \mathbb{M}_{\mathbb{C}} \longleftrightarrow \text{Riemann spheres in } \mathbb{PT}$$

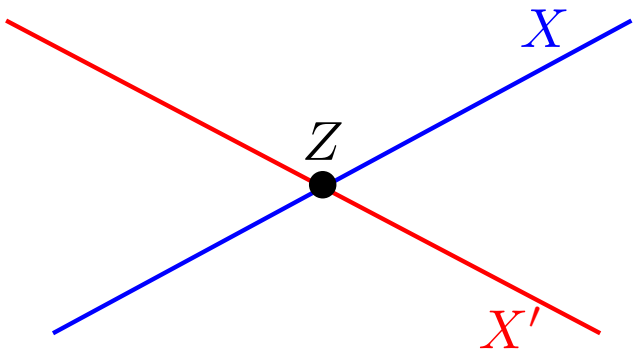
$$\text{Null 2-planes in } \mathbb{M}_{\mathbb{C}} \longleftrightarrow \text{Points in } \mathbb{PT}$$

The Visual Picture

Space-time:



Twistor Space:



Why Twistors?

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Key Question

Is there a natural *action principle* on $\mathbb{PT}$ that reduces to ASDYM theory and 4D Chern–Simons theory?

Holomorphic Chern–Simons Theory

Holomorphic Chern–Simons Theory on $\mathbb{PT}$

- **Holomorphic Chern–Simons (HCS) theory**[3], has an action defined as:

$$S_{\mathrm{HCS}}[\bar{\mathcal{A}}] = \frac{1}{2\pi i}\int_{\mathbb{PT}} \Omega \wedge \mathrm{HCS}(\bar{\mathcal{A}}),$$

$$\mathrm{HCS}(\bar{\mathcal{A}}) = \mathrm{tr}\left(\bar{\mathcal{A}} \wedge \bar{\partial}\bar{\mathcal{A}} + \tfrac{2}{3}\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right).$$

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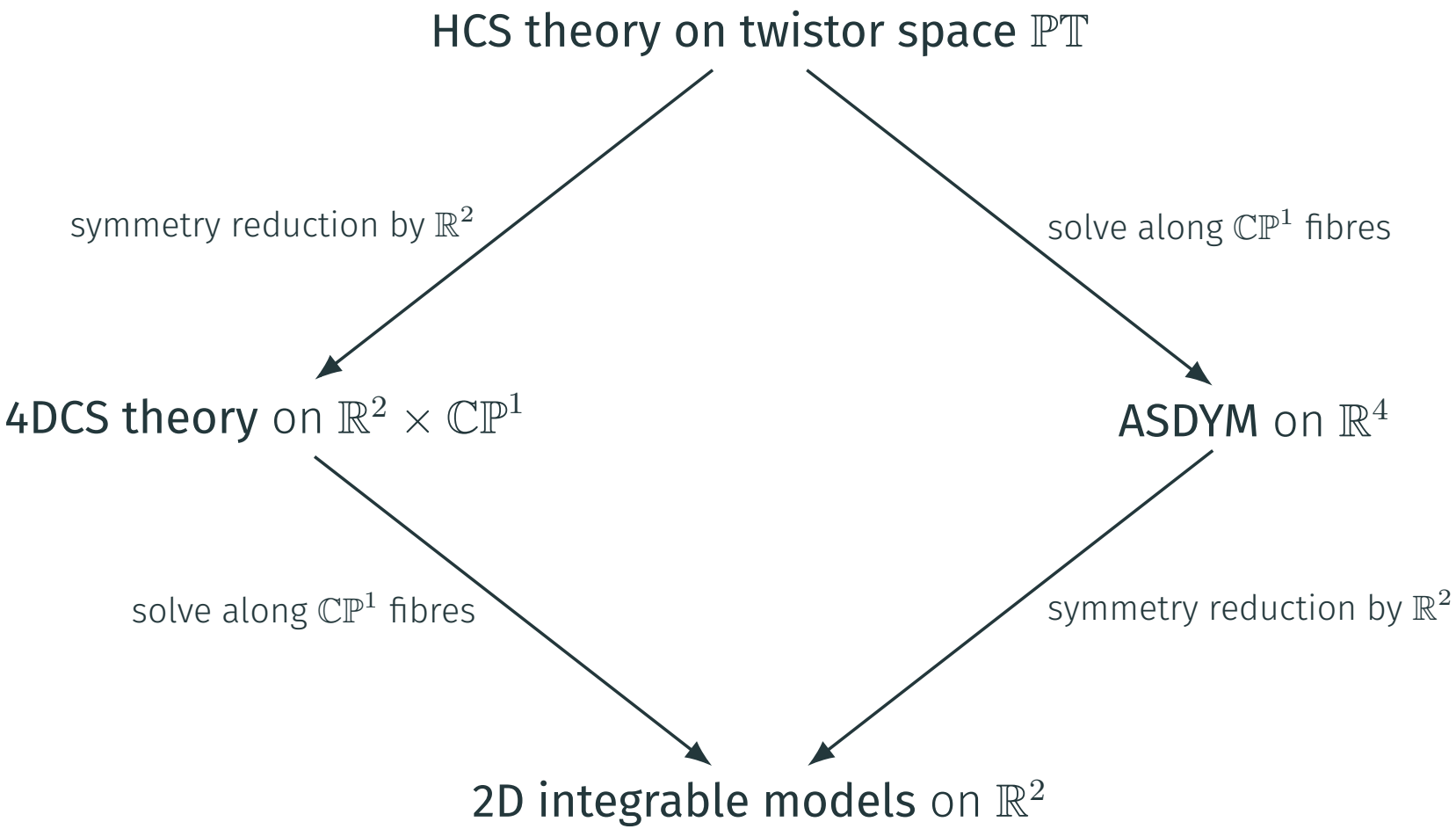
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- The equations of motion are simply

$$\bar{\mathcal{F}}(\bar{\mathcal{A}}) = \bar{\partial}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \bar{\mathcal{A}} = 0.$$

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The Full Picture



Conclusion

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- The **spectral parameter ζ is encoded** as the coordinate on the $\mathbb{CP}^1$ fibre of twistor space.
- HCS theory on $\mathbb{PT}$ serves as a (sort of) **master theory**: ASDYM, 4DCS theory, and 2D integrable models all descend from it via reduction.
- **Future work:** tracing explicit solutions through the chain and looking at broader classes of reductions.

Questions?