Higher Dimensional Integrability and Twistor Actions

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Introduction

Much of theoretical physics is governed by *non-linear* differential equations, and most resist analytic treatment. There is a rare subset of these non-linear systems called **integrable systems**. It is difficult to give a universal definition for integrability, but some features typical of "integrable" systems are:

1. The equations are to some degree solvable, i.e. many explicit solutions exist and there exist algorithms or methods for finding these solutions.
2. Nonlinear superpositions of solutions are possible.
3. The system is not chaotic in nature due to a large number of constants of motion.

Integrable systems often admit a **Lax pair**, consisting of two differential operators $L(\zeta)$, $M(\zeta)$ depending on a complex **spectral parameter** $\zeta \in \mathbb{CP}^1$, such that

$$[L(\zeta),\, M(\zeta)] = 0 \iff \text{equations of motion for all } \zeta.$$

Integrable equations look very different yet share the same structural features and solvability. Why?

Anti–Self–Dual Yang–Mills Theory

Yang–Mills theory consists of a gauge field A with curvature

$$F = dA + A \wedge A.$$

The **anti-self-duality condition** imposes

$$F = -\star F.$$

The ASDYM equation is integrable: it admits a **Lax pair**

$$L = D_w - \zeta\, D_{\tilde{z}}, \qquad M = D_z - \zeta\, D_{\tilde{w}},$$

where $\zeta \in \mathbb{CP}^1$. $[L, M] = 0$ for all $\zeta \iff$ ASDYM equations. Many 1+1D integrable models (sine-Gordon, KdV, NLS, PCM, ...) arise as *dimensional reductions* of ASDYM [1].

Chern–Simons Theories

Chern-Simons theory is a topological quantum field theory. The action is defined by:

$$S_{3\text{dCS}}[A] = \frac{k}{4\pi}\int_{M_3} \text{CS}[A], \quad \text{CS}[A] = \text{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right).$$

It is not strictly gauge-invariant, but its variation under a gauge transformation differs by quantised terms that will not contribute to the path integral. The equations of motion for 3dCS theory are given by:

$$F[A] = dA + A \wedge A = 0.$$

One can extend Chern-Simons to a four-dimensional theory in order to be able to work on a four-manifold $M_4 = C \times \Sigma$, where C is a complex curve and Σ is a 2D manifold. To do this, one must introduce a one-form ω to define a new action:

$$S_{4\text{dCS}}[A] = \frac{1}{2\pi i}\int_{M_4} \omega \wedge \text{CS}[A].$$

The precise conditions ω must satisfy are currently not known, so one can assume it is the pullback of a general meromorphic (1,0)-form [2]. Examples of such choices are:

$$d\zeta, \quad \frac{d\zeta}{\zeta}, \quad \frac{1-\zeta^2}{\zeta^2} d\zeta.$$

We can see the structure of these forms on a $\mathbb{CP}^1$ manifold in the figures below.

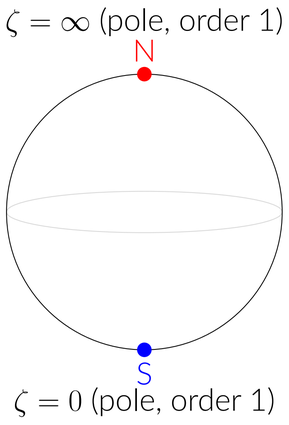


Figure 1. Two poles of $\omega = \frac{d\zeta}{\zeta}$ at $\zeta = 0$ (South) and $\zeta = \infty$ (North).

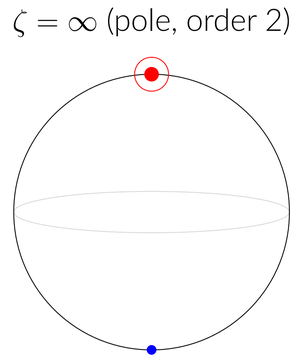


Figure 2. A double pole of $\omega = d\zeta$ at $\zeta = \infty$ and no poles or zeros at $\zeta = 0$.

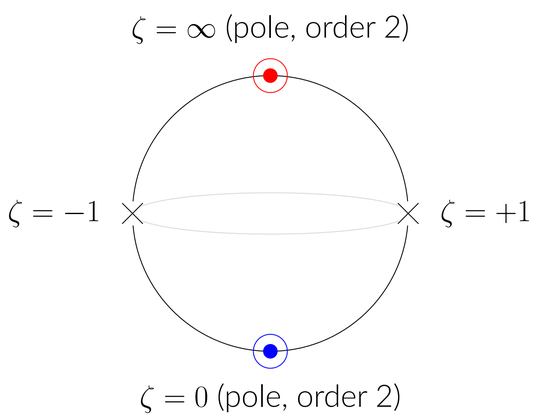


Figure 3. Two double poles of $\omega = \frac{(1-\zeta^2)}{\zeta^2} d\zeta$ at $\zeta = 0$ and $\zeta = \infty$, with simple zeros at $\zeta = \pm 1$.

4D Chern–Simons can also be dimensionally reduced to many 2D integrable models.

Twistor Theory

Twistor space $\mathbb{PT}$ is an open subset of $\mathbb{CP}^3$ [3], with homogeneous coordinates

$$Z^A = (\mu^{\dot{\alpha}},\, \lambda_\alpha), \quad Z^A \sim rZ^A, \quad r \in \mathbb{C}\backslash\{0\},$$

where $\mu^{\dot{\alpha}}$ and λ_α are two-component spinors of opposite chirality. For a point in complexified Minkowski space $x^{\alpha\dot{\alpha}} \in \mathbb{M}_\mathbb{C}$, the **incidence relation** $\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha$ encodes a non-local correspondence:

Points in $\mathbb{M}_\mathbb{C}$	$\longleftrightarrow$	Riemann spheres in $\mathbb{PT}$
Null 2-planes in $\mathbb{M}_\mathbb{C}$	$\longleftrightarrow$	Points in $\mathbb{PT}$

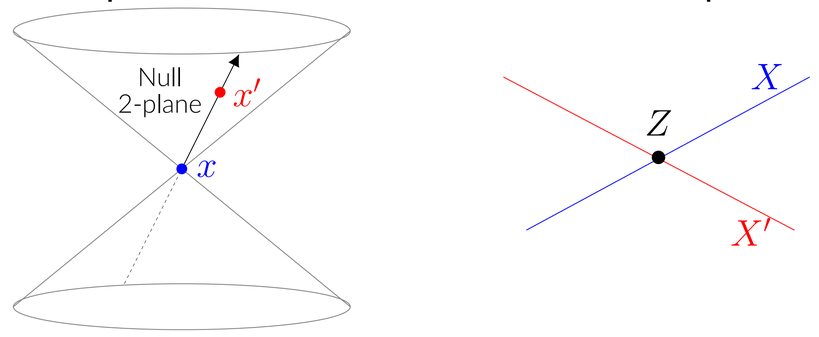


Figure 4. Null-separated points x, x' on light cone correspond to lines X, X' intersecting at $Z \in \mathbb{PT}$.

To recover real Minkowski space from this complex picture, we impose a reality condition on $\mathbb{PT}$. In Lorentzian signature, complex conjugation sends a twistor to a dual twistor,

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha) \mapsto \bar{Z}^A = (\bar{\lambda}^{\dot{\alpha}}, \bar{\mu}_\alpha),$$

inducing a natural inner product

$$Z \cdot \bar{Z} = [\mu\, \bar{\lambda}] + \langle \bar{\mu}\, \lambda \rangle.$$

A line $X \subset \mathbb{PT}$ corresponds to a point of real Minkowski space $\mathbb{M}$ if and only if

$$Z \cdot \bar{Z} = 0 \quad \text{for all } Z \in X,$$

i.e. the line lies in the space of **null twistors**

$$\mathbb{PN} = \{Z \in \mathbb{PT} | Z \cdot \bar{Z} = 0\}.$$

A point in $\mathbb{PN}$ corresponds to a real null geodesic in $\mathbb{M}$, the intersection of the null complex 2-plane with the slice $\mathbb{M} \subset \mathbb{M}_\mathbb{C}$.

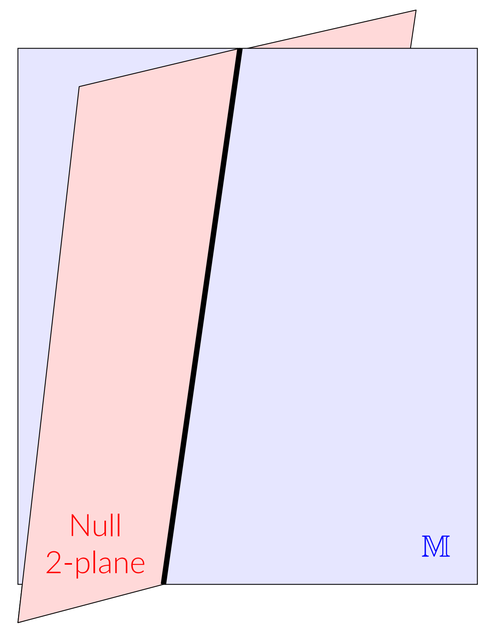


Figure 5. Intersection between null complex 2-plane and real Minkowski space $\mathbb{M} \subset \mathbb{M}_\mathbb{C}$.

Holomorphic Chern–Simons Theory on $\mathbb{PT}$

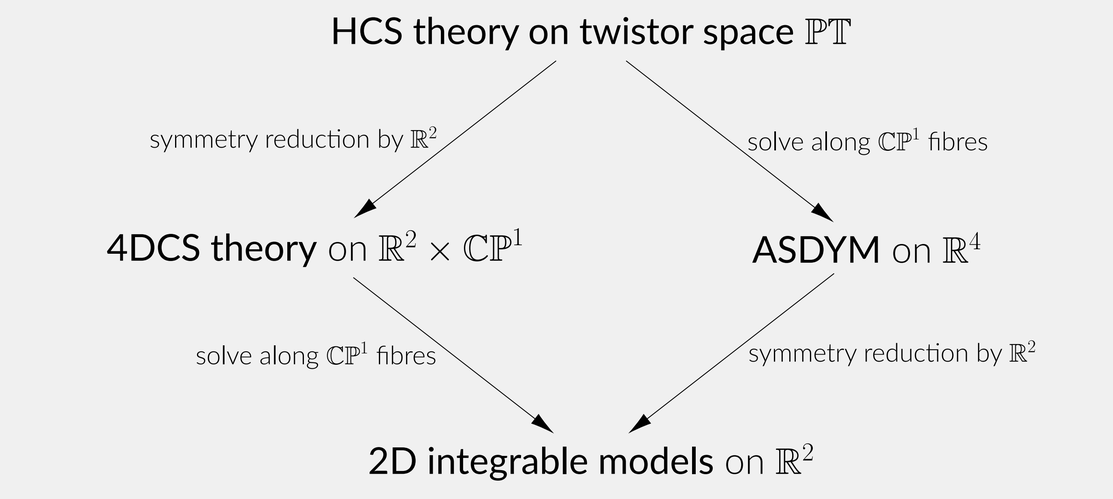
Holomorphic Chern–Simons (HCS) is a 6D field theory on twistor space with action

$$S_{\text{HCS}}[\bar{\mathcal{A}}] = \frac{1}{2\pi i}\int_{\mathbb{PT}} \Omega \wedge \text{tr}\left(\bar{\mathcal{A}} \wedge \bar{\partial}\bar{\mathcal{A}} + \tfrac{2}{3}\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right),$$

where Ω is a meromorphic $(3,0)$-form and $\bar{\mathcal{A}}$ is a $(0,1)$-form gauge field [4]. The equations of motion state that $\bar{\mathcal{A}}$ defines a **holomorphic structure** on the bundle:

$$\bar{\mathcal{F}}(\bar{\mathcal{A}}) = \bar{\partial}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \bar{\mathcal{A}} = 0.$$

HCS on $\mathbb{PT}$ can then be dimensionally reduced to get different integrable models.



Conclusion/Outlook

- The **spectral parameter** ζ is the coordinate on the $\mathbb{CP}^1$ fibre of $\mathbb{PT}$, and has a geometric origin rather than just being an auxiliary parameter.
- HCS encodes ASDYM, 4D CS, and many 2D integrable models simultaneously via two commuting reduction routes.
- **Future work:** tracing explicit solutions through the chain and looking at broader classes of reductions.

References

[1] Lionel J. Mason and Nicholas M. J. Woodhouse. *Integrability, Self-Duality, and Twistor Theory*. Number 15 in London Mathematical Society Monographs. Oxford University Press, Oxford, 1996.
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[4] Roland Bittleston and David Skinner. Twistors, the ASD Yang–Mills equations and 4D Chern–Simons theory. *Journal of High Energy Physics*, 2023(2):227, 2023. arXiv:2011.04638 [hep-th].