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Capstone Project Report

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Higher-dimensional Integrability and Twistor Actions

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Declaration

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Abstract

Integrable systems occupy a special role in theoretical physics. Unlike many non-linear differential equations, integrable equations admit analytic solutions, solving algorithms, and many conserved quantities. Understanding integrable equations, and whether they all share a common higher dimensional origin, is an open problem. This work investigates holomorphic Chern–Simons theory and its role as a candidate unified origin for integrability, following the initial work of Costello, Witten, and Yamazaki and further developed by Bittleston and Skinner.

We present in detail the dimensional reduction of holomorphic Chern–Simons theory on twistor space, showing explicitly at the level of the Lagrangian how it descends to both the 4d Wess–Zumino–Witten model and 4d Chern–Simons theory. The WZW_4 model has Yang's equation as its equation of motion, which is precisely the on-shell condition of the anti-self-dual Yang–Mills equation; the two theories are thus classically equivalent on-shell via this shared equation of motion. The principal chiral model appears as a common further reduction of both.

We then attempt to carry explicit solutions through the dimensional reduction chain using the Atiyah–Ward ansatz. As an original contribution, it is shown that no non-trivial solution of this ansatz type is compatible with the translation invariance required for the symmetry reduction, ruling out the entire class from the Bittleston–Skinner correspondence. The BPST instanton is examined explicitly as an illustration of this obstruction.

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"Using a term like non-linear science is like referring to the bulk of zoology as the study of non-elephant animals."

— Stanislaw Ulam

1. Introduction

Much of theoretical physics is governed by non-linear differential equations, which play a central role in describing complex physical phenomena. Unlike linear systems, where general solution techniques are readily available, non-linear equations typically resist analytic treatment. Unfortunately, the majority of reality is described by non-linear equations. Due to this, most progress in physics arises either through approximation methods, such as perturbation theory, or by exploiting large symmetry reductions. Yet some rare non-linear systems are exactly solvable, and these are called *integrable* systems. These special systems admit infinitely many conserved quantities, Lax pair formulations, and often deep geometric structures.

Self-dual Yang–Mills (SDYM) theory provides a four-dimensional origin for many such integrable models. When the Yang–Mills field strength obeys the self-duality condition, the theory becomes integrable and admits powerful solution-generating techniques. Many well-known 1+1–dimensional integrable models, including the sine-Gordon and KdV equations, arise as dimensional reductions of SDYM [1]. This raises a fundamental question: *What underlying structure explains the integrability of these diverse systems, and can we identify a higher-dimensional "master" theory from which they all descend?*

Twistor theory offers a compelling answer. In this framework, the spectral parameter of an integrable system is identified with a complex coordinate on a Riemann sphere, viewed as a $\mathbb{CP}^1$ fiber over space-time. Twistor methods, through the Penrose–Ward transform, establish a correspondence between self-dual gauge fields and holomorphic vector bundles over twistor space. This geometric correspondence provides a method for constructing exact solutions and offers a unified geometric picture underlying many integrable models in $1+1$ and $2+1$ dimensions.

An alternative picture of integrable models originating from four–dimensional Chern–Simons theory was provided in [2]. This is in fact closely related to the twistor space description and it was shown in [3, 4], that one can define a six–dimensional holomorphic Chern–Simons theory on twistor space that is related to the four–dimensional Chern–Simons theory and that also reproduces the so-called Yang equations of self-dual Yang–Mills. This suggests that 6dHCS may provide a unified framework for understanding integrability in a wide range of physical models. Although progress on this front is still developing, the possibility of viewing many integrable equations as part of a single twistor-based structure is both conceptually striking and mathematically rich, motivating further investigation.

2. Background

2.1 Integrability

Integrability is hard to define universally, hence it is easier to give examples than a general definition [1].

Examples of Integrability

For instance, $2n$-dimensional Hamiltonian systems are called integrable if you can find n constants of the motion in involution. Constants of the motion in involution mean quantities that commute with the Hamiltonian, i.e. are time-invariant. Not only this, they must also commute with each other. Since these quantities commute with each other as well as the Hamiltonian, they can also be used as a new notion of a "time" parameter, so one can evolve the system with these parameters, and the other constants of motion in involution will be invariant. For systems of partial differential equations, there are infinitely many degrees of freedom, so the definition is less clear. There are still some features that are typical of systems that are considered "integrable":

1. The equations are to some degree soluble, i.e. many explicit solutions exist and there exist algorithms or methods for finding these solutions.
2. Nonlinear superpositions of solutions are possible.
3. The system is not chaotic in nature due to a large number of constants of motion.
4. They have the Painleve property, (solutions have only poles as movable singularities).

The Euler top is a classic finite-dimensional integrable system. This is the dynamical evolution of an asymmetric body spinning about its centre of mass, and is given by the geodesic flow of a left-invariant metric on SO(3). Geodesics can be thought of as straight lines in curved space. A left invariant metric is one which is unaffected by acting on both vectors from the left, and is therefore completely determined by an inner product on the Lie algebra $\mathfrak{g} = T_eG$. This means that the geometry of the Lie group, in this case SO(3), is the same wherever you are on the manifold. Two other top configurations are also integrable:

1. **The Lagrange top:** In this case, the fixed point and the centre of gravity lies on the axis of symmetry.
2. **The Kovalevskaya top:** For some constant $A > 0$, this has principal moments at $2A$, $2A$ and A at the fixed point, with the centre of mass lying in the principal axes of moment $2A$.

The Korteweg—de Vries (KdV) equation is another famous integrable equation, which is given by Eq. (2.1).

$$4u_t - u_{xxx} - 6uu_x = 0. \tag{2.1}$$

This equation governs shallow water waves, and a solitary wave is described by what is known as a soliton solution to this equation:

$$u = 2\kappa^2 \mathrm{sech}^2(\kappa x + \kappa^3 t - \kappa x_o), \text{ where } x_0 \text{ and } \kappa \text{ are constants.} \tag{2.2}$$

A soliton is a wave that continues to propagate at a constant velocity, and arises here as t increases due to the balance between the dissipative term u_{xxx} and the nonlinear term $6uu_x$. The integrability of this equation can be traced to the existence of a Lax pair, i.e. two differential operators

L and M that commute with each other. In this case

$$L = \partial_x^2 + u, \quad \text{and} \quad M = \partial_t - \partial_x^3 - \frac{3}{2}u\partial_x - \frac{3}{4}u_x. \tag{2.3}$$

This alone doesn't characterize integrability, but most methods for solving integrable systems start with this. An example of one of these methods is the inverse-scattering transform. Other soliton equations with similar features include the **nonlinear Schrodinger equation**,

$$i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi \pm |\psi|^2\psi, \tag{2.4}$$

and the **sine-Gordon equation**,

$$\partial_t^2\phi - \partial_x^2\phi = \sin\phi. \tag{2.5}$$

Another example of an integrable equation is **the anti-self-dual Yang–Mills (ASDYM) equation**, which is the condition that the curvature tensor $F_{\mu\nu}$ defined in Eq. (2.29) should be anti-self-dual, i.e.,

$$F_{\mu\nu} = -*F_{\mu\nu}. \tag{2.6}$$

Here, the Hodge star operator $*$ is defined as the following for a general p-form β in n dimensions:

$$*\beta_{ab\ldots c} = \frac{1}{(n-p)!}\Delta\epsilon_{ab\ldots c}{}^{de\ldots f}\beta_{de\ldots f}, \quad \Delta^2 = \det(\eta), \tag{2.7}$$

where η is the metric tensor. There is a natural linear system since anti-self-duality is the integrability condition for the existence of covariantly constant sections of the vector bundle over totally null 2-planes with self-dual tangent bivectors. A vector bundle is essentially a family of vector spaces smoothly attached to each point of a manifold. An example is attaching a tangent space T_pM at each point p of a manifold M. Covariantly constant sections of these vector bundles are vectors s in the vector spaces of the vector bundle which satify the following condition:

$$Ds = 0, \quad D = \mathrm{d} + iA. \tag{2.8}$$

A totally null 2-plane is a 2-dimensional subspace of the tangent space on which the metric restricts to zero, i.e. $\eta(u, v) = 0$ for all vectors u, v in the plane. In Euclidean signature this forces $u = v = 0$ since the metric is positive definite, which is why we must work in complex or indefinite signatures. In ultrahyperbolic signature $(+, +, -, -)$ for instance, the vectors $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$ span a totally null 2-plane. A tangent bivector is a wedge product of two tangent vectors $v\wedge w$, representing the oriented area element of the 2-plane they span. A self-dual tangent bivector is one preserved by the Hodge star,

$$*(v \wedge w) = v \wedge w. \tag{2.9}$$

This interpretation is the basis of what we call the Penrose–Ward transform, solutions of which are parametrized by holomorphic vector bundles over parts of twistor space $\mathbb{CP}^3$. An "integrable property" of the ASDYM equation is a four-dimensional version of the Painleve property. **The Painleve equations** are nonlinear ODEs with special transcendatal solutions, which have both fixed and movable singularities, but the movable singularities are all poles. There is a connection between these equations and the isomonodromy problem. Another example of equations with this Painleve property are the generalized Riccati equations:

$$y' = a(t)y^2 + b(t)y + c(t). \tag{2.10}$$

Some equations are integrable simply because they can be transformed to linear equations. An example is **Burgers' equation**

$$u_w = 2uu_z + u_{zz}, \tag{2.11}$$

which is linearised by the substitution $u = \phi_z/\phi$, reducing it to the heat equation

$$\phi_w - \phi_{zz} = 0. \tag{2.12}$$

This is included as a contrasting example: while Burgers' equation is solvable, it is integrable in a more trivial sense than the KdV or ASDYM equations, since no nonlocal transform such as inverse scattering is needed. This distinction motivates looking for deeper geometric structure in the more interesting cases.

2.2 The Anti-Self-Dual Yang–Mills Equation

The material in this section closely follows [1], the definitive reference connecting ASDYM integrability to twistor theory. We develop the necessary mathematical background before deriving key formulations.

2.2.1 Mathematical Background

Gauge Theories

The Yang–Mills equations are PDEs in 4d. There is a system of equations for each type of Lie group G, called a gauge group. In the special case where $G = U(1)$, they reduce to Maxwell's equations. The equations are non-linear for a non-Abelian gauge group, such as $SU(2)$. To begin considering more general theories than electromagnetism, one must first begin to consider more fundamental objects than electric and magnetic fields, such as the 4-potential A. The second thing to do is to consider A as a connection one-form, which allows us to define a notion of a covariant derivative acting on wave functions taking values in a complex line bundle. A complex line bundle is a smooth assignment of a complex one-dimensional vector space to every point of a manifold. The precise notion of a connection one-form and its associated curvature will be defined carefully below. In four dimensions, a curvature 2-form F can be split into self-dual and anti-self-dual parts. These correspond to splitting the electromagnetic fields into a superposition of $\mathbf{B} = i\mathbf{E}$ (SD) and $\mathbf{B} = -i\mathbf{E}$ (ASD). Every connection with self-dual curvature automatically satisfies the Yang–Mills equations, and unlike in general this self-dual case is integrable.

Space-Time

In Lorentzian signature $(+---)$, there are no real nontrivial self-dual or anti-self-dual Maxwell fields as the condition $\mathbf{B} = \pm i\mathbf{E}$ is incompatible with the reality of electromagnetic fields. This is why we consider Euclidean $(++++)$ and ultrahyperbolic $(++--)$ signatures, where the reality condition does not force a self-dual and anti-self-dual 2-form to vanish. These signatures are not directly physical, but are often useful. It is like performing a Wick rotation in QFT (Euclidean), which is used to compute quantities, before analytically continuing back to Minkowskian signature. Ultrahyperbolic signature is like Wick rotating one of the spatial components instead of the time component. It is useful to deal with all of these in one framework, so they are embedded in complex Minkowski space $\mathbb{M}_{\mathbb{C}}$. In double null coordinates, the metric and volume element are given by:

$$\mathrm{d}s^2 = 2(\mathrm{d}z\,\mathrm{d}\tilde{z} - \mathrm{d}w\,\mathrm{d}\tilde{w}), \quad \nu = \mathrm{d}w \wedge \mathrm{d}\tilde{w} \wedge \mathrm{d}z \wedge \mathrm{d}\tilde{z}. \tag{2.13}$$

The coordinate vectors ∂_w, ∂_z, $\partial_{\tilde{w}}$, $\partial_{\tilde{z}}$ form a null tetrad at each point of space-time. A general null tetrad is a basis of 4-vectors $\{W, Z, \tilde{W}, \tilde{Z}\}$ such that

$$\eta(Z, \tilde{Z}) = -\eta(W, \tilde{W}) = 1, \quad 24\nu(W, \tilde{W}, Z, \tilde{Z}) = 1, \tag{2.14}$$

where η is the metric tensor, and such that all the other inner products vanish. The different "real slices" are obtained by imposing the following reality conditions on w, $\tilde{w}$, z, $\tilde{z}$:

$$\begin{aligned}
\mathbb{R}^4 &: \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & -x^2 + ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}, \\
\mathbb{M} &: \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - x^1 \end{pmatrix}, \\
\mathbb{U} &: \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}.
\end{aligned} \tag{2.15}$$

A real slice of these spaces is essentially obtained by restricting to coordinates satisfying a reality condition; for instance, one real slice of $\mathbb{C}$ would be the real line, though other real slices are possible depending on the condition imposed. Just because we impose reality conditions to take a real slice doesn't mean there isn't utility in complexifying our space to start off, as many things are just easier when this is done, e.g. residue integrals. While all solutions obtained in this complexified

space may not be real or physical, they can still be of mathematical interest. On $\mathbb{R}^4$ and $\mathbb{U}$, the volume form ν defined above is real and coincides with

$$\mathrm{d}^4x = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{2.16}$$

in Cartesian coordinates, however the same volume form is always imaginary on $\mathbb{M}$. A general coordinate system on $\mathbb{M}_{\mathbb{C}}$ shall be denoted by coordinates x^a $(a = 0, 1, 2, 3)$, with the following definitions:

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b, \quad \nu = \nu_{abcd}\,\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d, \quad \nu_{abcd} = \frac{1}{24}\Delta\epsilon_{abcd}, \quad \Delta^2 = \det(\eta_{ab}). \tag{2.17}$$

Differential Forms

A p-form β has components given by a skew-symmetric covariant tensor $\beta_{ab\ldots c}$ such that

$$\beta(X, Y, ..., Z) = \beta_{ab\ldots c}X^aY^b \cdots Z^c \tag{2.18}$$

for arbitrary vector fields $X, Y, ..., Z$. The exterior derivative $d\beta$ has components

$$\partial_{[a}\beta_{bc\ldots d]}, \tag{2.19}$$

and the wedge product $\beta \wedge \gamma$, where γ is a q-form is a $p + q$ form with components

$$\beta_{[ab\ldots c}\gamma_{de\ldots f]}. \tag{2.20}$$

Here square brackets denote total antisymmetrization over the enclosed indices, for example,

$$A_{[ab]} = \frac{1}{2}\left(A_{ab} - A_{ba}\right). \tag{2.21}$$

With these definitions, the Lie derivative for a vector field X is given by

$$\mathcal{L}_X\beta = X \lrcorner\, \mathrm{d}\beta + \mathrm{d}(X \lrcorner\, \beta), \tag{2.22}$$

where

$$(X \lrcorner\, \gamma)_{bc\ldots d} = qX^a\gamma_{ab\ldots d}. \tag{2.23}$$

The Hodge star operator in n-dimensions maps p-forms to $(n - p)$-forms, and was defined earlier. For 2-forms in 4d, the Hodge dual is also a 2-form and given by

$$\beta^*_{ab} = \frac{1}{2}\Delta\epsilon_{abcd}\eta^{ce}\eta^{df}\beta_{ef}. \tag{2.24}$$

Also, $*^2 = 1$, thus the space of 2-forms decomposes into the direct sum of eigenspaces of * with eigenvalues ± 1. We say that β is self-dual (SD) whenever $*\beta = \beta$, or anti-self-dual (ASD) whenever $*\beta = -\beta$, as was mentioned earlier. The forms

$$\alpha = \mathrm{d}w \wedge \mathrm{d}z, \quad \tilde{\alpha} = \mathrm{d}\tilde{w} \wedge \mathrm{d}\tilde{z}, \quad \text{and} \quad \omega = \mathrm{d}w \wedge \mathrm{d}\tilde{w} - \mathrm{d}z \wedge \mathrm{d}\tilde{z} \tag{2.25}$$

span the space of SD 2-forms; and

$$\mathrm{d}w \wedge \mathrm{d}\tilde{z}, \quad \mathrm{d}\tilde{w} \wedge \mathrm{d}z, \quad \text{and} \quad \mathrm{d}w \wedge \mathrm{d}\tilde{w} + \mathrm{d}z \wedge \mathrm{d}\tilde{z} \tag{2.26}$$

span the space of ASD 2-forms.

Conformal Transformations and Compactified Space-Time

Conformal transformations are especially interesting in four dimensions because they preserve the tensor field $\nu_{ab}{}^{cd}$ and hence the duality operator on 2-forms. Therefore the ASD condition on the curvature of a connection is conformally invariant. At the infinitesimal level, these conformal transformations are given by conformal Killing vectors, i.e., vector fields K that satisfy the following equation:

$$\partial_{(a}K_{b)} = \frac{1}{4}\eta_{ab}\partial_c K^c. \tag{2.27}$$

The parentheses mean total symmetrization instead of total antisymmetrization. A Killing vector can be thought of as a flow of coordinates that leaves the metric invariant. For constant metrics like Euclidean or Minkowski this can be achieved by any translation by a constant, but sometimes metrics depend on coordinates, like the Schwarzchild metric. Equivalently, it is the condition that $\mathcal{L}_K \eta_{ab} \propto \eta_{ab}$, or that the flow of K should be a one-parameter family of conformal transformations. When $\partial_c K^c = 0$, K is a Killing vector and the transformations are isometries. An isometry is a distance-preserving transformation between metric spaces, such as translations, rotations and reflections in Euclidean spaces. The space of conformal Killing vectors is 15 dimensional, and in general is given by:

$$K_a = T_a + L_{ab}x^b + Rx_a + x^b x_b S_a - 2S_b x^b x_a. \tag{2.28}$$

The components of T label the translations (four parameters), the components of L label the rotations and Lorentz transformations (six parameters), R (one parameter) labels the dilatations, and the components of S label the special conformal transformations (four parameters). For more details, see [5]. The complex conformal group is isomorphic to the projective general linear group $\mathrm{PGL}(4,\mathbb{C}) = \mathrm{GL}(4,\mathbb{C})/\mathbb{C}^\times$. This is done by a construction that is central to twistor theory, where ${\mathbb{M}_\mathbb{C}}^\#$ is identified with the Klein quartic in $\mathbb{CP}^5$.

Bundles, Connections, and Curvature

A vector bundle E is a collection of vector spaces E_x attached to each point x in a manifold M, for example the tangent bundle TM is a manifold of the tangent planes T_pM generated at each point $p \in M$. These vector spaces E_x vary smoothly or holomorphically with x depending on if M is a real or complex manifold. More precisely, it is a manifold E (the total space) together with a projection map $\pi : E \to M$, such that each fibre $\pi^{-1}(x)$ has the structure of an n-dimensional vector space. A local section of E is a map $s : U \subset M \to E$ such that $\pi(s(x)) = x$ for every $x \in U \subset M$, and it is global if $U = M$. A local trivialization is a set of sections $e_1, \ldots, e_n$ such that $\{e_i(x)\}$ is a basis for E_x for any $x \in U \subset M$. Two local trivializations or local frame fields, defined on overlapping open sets U and V with $U \cap V \neq \emptyset$, are related by $\tilde{e}_j = e_i g_{ij}$, where g_{ij} are known as transition functions or patching matrices. If there is some additional structure in the bundle, these g_{ij}'s can belong to a subgroup of the general linear group. For instance, if each fibre has a Hermitian metric on it, these transition functions will take values in $\mathrm{U}(n)$. These subgroups are called the structure group of the bundle (if there is no additional structure the structure group is simply $\mathrm{GL}(n)$).

Having described the algebraic structure of vector bundles via transition functions and structure groups, we now introduce the key differential-geometric ingredient: a connection on the bundle. A connection $D = \mathrm{d} + A$ provides a notion of parallel transport across the bundle, where $A = A_a \mathrm{d}x^a$ is a Lie algebra-valued one-form known as the gauge field or connection one-form. The curvature of the connection is given by

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]. \tag{2.29}$$

It measures the extent to which D_a and D_b fail to commute, since

$$[D_a, D_b]s = F_{ab}s \tag{2.30}$$

for any section s. The curvature F can also be written as

$$F = \mathrm{d}A + A \wedge A. \tag{2.31}$$

Naturally, transformations of the gauge field A also affect the curvature F since it is built from it, unless the gauge field is Abelian like U(1) for electromagnetism.

The Yang–Mills Equations

The simplest gauge theory is that of the interaction between a classical electromagnetic field and a complex wave function. The gauge group here is U(1), so gauge transforms don't affect

the curvature tensor $F_{\mu\nu}$. A gauge transformation transforms the covariant derivative and the wavefunction as follows:

$$\begin{aligned} A_\mu \to \tilde{A}_\mu &= A_\mu + \partial_\mu \phi \\ \varphi \to \tilde{\varphi} &= e^{-i\phi}\varphi \\ (\partial_\mu + i\tilde{A}_\mu)\tilde{\varphi} &= (\partial_\mu + iA_\mu + i\partial_\mu\phi)(e^{-i\phi}\varphi) \\ &= e^{-i\phi}(\partial_\mu + iA_\mu)\varphi + i\partial_\mu\phi e^{-i\phi}\varphi - i\partial_\mu\phi e^{-\phi}\varphi \\ &= e^{-i\phi}(\partial_\mu + iA_\mu)\varphi. \end{aligned} \tag{2.32}$$

Hence the covariant derivative is invariant under this gauge transformation up to a phase factor. In terms of electric and magnetic fields, $F = F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is given by

$$F_{\mu\nu} = i\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{2.33}$$

The i prefactor is a convention, and it is done here because without the i there is no SD condition on F. $-i$ is a different orientation. In the absence of sources, Maxwell's equations are given by

$$\begin{aligned} \mathrm{d}F = 0: &\quad \nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \mathbf{0}, \\ \mathrm{d}*F = 0: &\quad \nabla\cdot\mathbf{E} = 0, \quad \nabla\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \mathbf{0}. \end{aligned} \tag{2.34}$$

In a general gauge theory, these are replaced by the Yang Mills Equations

$$D_{[a}F_{bc]} = 0, \quad D^a F_{ab} = 0. \tag{2.35}$$

The first equation is the Bianchi Identity, while the second is the Euler-Lagrange equation of the Lagrangian density $\mathcal{L} = \frac{1}{4}\mathrm{tr}(F_{ab}F^{ab})$. For a non-Abelian gauge field, these equations are non-linear, but a self-dual or anti-self dual curvature tensor automatically satisfies these equations and makes them integrable.

2.2.2 ASD Electromagnetic fields

Having established the necessary mathematical framework, including the geometry of vector bundles, connections, and curvature, as well as the conformal structure of complexified Minkowski space, we are now in a position to study the anti-self-dual Yang–Mills equation in detail, beginning with the electromagnetic case before turning to the full non-Abelian theory. The ASD equation for the curvature is simply

$$F = -*F. \tag{2.36}$$

An equivalent condition is that the curvature should vanish on restriction to α-planes, as the α-planes have self-dual tangent bivectors, and the contraction of this bivector with an anti-self-dual curvature will be zero. A solution to the ASDYM equation necessarily satisfies the Yang–Mills equations, as the Bianchi identity $DF = 0$ holds for any connection, and since $F = -*F$ then $D*F = 0$ also.

In the electromagnetic case in Minkowski space, substituting $\mathbf{B} = -i\mathbf{E}$ into Maxwell's equations eliminates $\mathbf{B}$ entirely, and the equations reduce to:

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\times\mathbf{E} - i\frac{\partial\mathbf{E}}{\partial t} = \mathbf{0}. \tag{2.37}$$

The solutions are necessarily complex, but can be interpreted as circularly polarized photon wave functions. The solutions are real in Euclidean and Ultrahyperbolic signatures. This can be seen by replacing t by $-it$ in the above equation to take the Euclidean case.

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\times\mathbf{E} + \frac{\partial\mathbf{E}}{\partial t} = 0. \tag{2.38}$$

In double null coordinates, the ASD condition is

$$\begin{aligned} &\partial_w A_z - \partial_z A_w = 0, \quad \partial_{\tilde{w}} A_{\tilde{z}} - \partial_{\tilde{z}} A_{\tilde{w}} = 0, \\ &\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z - \partial_w A_{\tilde{w}} + \partial_{\tilde{w}} A_w = 0, \end{aligned} \tag{2.39}$$

where $A = A_w \mathrm{d}w + A_z \mathrm{d}z + A_{\tilde{w}} \mathrm{d}\tilde{w} + A_{\tilde{z}} \mathrm{d}\tilde{z}$. The first two equations are interpreted as integrability conditions, which imply the existence of functions u and v such that

$$A = \partial_w u \, \mathrm{d}w + \partial_z u \, \mathrm{d}z + \partial_{\tilde{w}} v \, \mathrm{d}\tilde{w} + \partial_{\tilde{z}} v \, \mathrm{d}\tilde{z}. \tag{2.40}$$

Since we can replace A with $A - \mathrm{d}u$, we set $A_z = A_w = 0$, and get that $A = \partial_{\tilde{w}} v \, \mathrm{d}\tilde{w} + \partial_{\tilde{z}} v \, \mathrm{d}\tilde{z}$. The second-order potential v is then defined up to the following:

$$v \mapsto v + f(w, z) + \tilde{f}(\tilde{w}, \tilde{z}), \tag{2.41}$$

where f and $\tilde{f}$ are arbitrary functions. This all gives us the following ASD condition for the function v:

$$\frac{\partial^2 v}{\partial z \partial \tilde{z}} - \frac{\partial^2 v}{\partial w \partial \tilde{w}} = 0. \tag{2.42}$$

When $\tilde{w} = -\bar{w}$, $\tilde{z} = \bar{z}$, this is the four-dimensional Laplace equation; and when $\tilde{w} = \bar{w}$, $\tilde{z} = \bar{z}$, or alternatively when $(w, z, \tilde{w}, \tilde{z})$ are all real, it is the "ultrahyperbolic" wave equation on $\mathbb{U}$. For the Euclidean case, an imaginary solution to Laplace's equation corresponds to an real one for the ASD condition in $\mathbb{R}^4$. A real solution to the ultrahyperbolic wave equation gives a real ASD 2-form in $\mathbb{U}$.

2.2.3 Lax Pairs

Now we wish to consider the general case for the ASDYM equation. If $D = \mathrm{d} + A$, then $F = F_{ab} \mathrm{d}x^a \wedge \mathrm{d}x^b$, where

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b] \, . \tag{2.43}$$

Again in double null coordinates, the ASD condition becomes:

$$\begin{aligned} &\partial_z A_w - \partial_w A_z + [A_z, A_w] = 0, \\ &\partial_{\tilde{z}} A_{\tilde{w}} - \partial_{\tilde{w}} A_{\tilde{z}} + [A_{\tilde{z}}, A_{\tilde{w}}] = 0, \\ &\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z - \partial_w A_{\tilde{w}} + \partial_{\tilde{w}} A_w + [A_z, A_{\tilde{z}}] - [A_w, A_{\tilde{w}}] = 0. \end{aligned} \tag{2.44}$$

The covariant derivative components are denoted by:

$$D_i = \partial_i + A_i, \quad i = w, z, \tilde{w}, \tilde{z}. \tag{2.45}$$

This gives us the equivalent relations:

$$[D_z, D_w] = 0, \quad [D_{\tilde{z}}, D_{\tilde{w}}] = 0, \quad [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0. \tag{2.46}$$

Another equivalent condition is that the Lax pair of operators

$$L = D_w - \zeta D_{\tilde{z}}, \quad M = D_z - \zeta D_{\tilde{w}} \tag{2.47}$$

should commute for every value of ζ, known as the spectral parameter.

2.2.4 Yang's Equation and the K-Matrix

Having established the Lax pair formulation of the ASDYM equation, we now seek to encode the gauge field in terms of simpler matrix-valued objects. This leads naturally to the introduction of Yang's matrix J and the K-matrix, which provide alternative but equivalent descriptions of ASD connections and will prove useful for deriving Lagrangian and Hamiltonian formulations. The first two equations (2.44) are the local integrability conditions for the existence of the matrix valued equations h and $\tilde{h}$ such that

$$\begin{aligned} &\partial_w h + A_w h = 0, \quad \partial_z h + A_z h = 0, \\ &\partial_{\tilde{w}} \tilde{h} + A_{\tilde{w}} \tilde{h} = 0, \quad \partial_{\tilde{z}} \tilde{h} + A_{\tilde{z}} \tilde{h} = 0. \end{aligned} \tag{2.48}$$

They are uniquely determined by A up to $h \mapsto h\tilde{P}$, $\tilde{h} \mapsto \tilde{h}P$, where $\tilde{P}$ is only a function of $\tilde{z}$ and $\tilde{w}$ and P is only a function of z and w. If A is replaced by the gauge equivalent potential $g^{-1}Ag+g^{-1}\mathrm{d}g$, then $h \mapsto g^{-1}h$ and $\tilde{h} \mapsto g^{-1}\tilde{h}$, which leaves $\tilde{h}^{-1}h$ invariant. This matrix $J = \tilde{h}^{-1}h$ is Yang's matrix. It is determined by D up to the freedom $J \mapsto P^{-1}J\tilde{P}$ and it determined D since A is equivalent to

$$J^{-1}\tilde{\partial}J = J^{-1}\partial_{\tilde{w}}J\mathrm{d}\tilde{w} + J^{-1}\partial_{\tilde{z}}J\mathrm{d}\tilde{z} \tag{2.49}$$

by the gauge transformation $A \mapsto h^{-1}Ah + h^{-1}\mathrm{d}h$. The first two equations (2.44) are satisfied by (2.49), the third holds if and only if J satisfies Yang's equation:

$$\partial_w(J^{-1}\partial_{\tilde{w}}J) - \partial_z(J^{-1}\partial_{\tilde{z}}J) = 0. \tag{2.50}$$

Yang's equation is equivalent to the ASD equations, but is not covariant under coordinate transformations that change the 2-planes spanned by (∂_z, ∂_w) and $(\partial_{\tilde{z}}, \partial_{\tilde{w}})$. It can also be written in the form

$$\partial(J^{-1}\tilde{\partial}J) \wedge \omega = 0, \tag{2.51}$$

where $\omega = \mathrm{d}w \wedge \mathrm{d}\tilde{w} - \mathrm{d}z \wedge \mathrm{d}\tilde{z}$, $\partial = \mathrm{d}w\,\partial_w + \mathrm{d}z\,\partial_z$ and $\tilde{\partial} = \mathrm{d}\tilde{w}\,\partial_{\tilde{w}} + \mathrm{d}\tilde{z}\,\partial_{\tilde{z}}$. Writing Yang's equation in general linear coordinates, we have

$$\partial_a(J^{-1}\partial^a J) + 2\omega^{ab}\partial_a(J^{-1}\partial_b J) = 0, \tag{2.52}$$

from which it can be determined that the equation is invariant only under conformal isometries that preserve ω^{ab} up to scale. The equation is covariant in the U(1) case ($J = e^{iv}$). The construction of J from a more geometric point of view can be understood by making the gauge transformations $g = h$ (potential has vanishing w and z components) and $g = \tilde{h}$ (potential has vanishing $\tilde{w}$ and $\tilde{z}$ components). The frame field $\{e_1, \ldots, e_n\}$ is chosen to be covariantly constant with respect to the ∂-directions, while $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ is covariantly constant with respect to the $\tilde{\partial}$-directions, meaning

$$\begin{aligned} D_w e_i = 0, \quad D_z e_i = 0, \\ D_{\tilde{w}}\tilde{e}_i = 0, \quad D_{\tilde{z}}\tilde{e}_i = 0, \end{aligned} \tag{2.53}$$

for $i = 1, \ldots, n$. The two frames are related by, $e_j = \tilde{e}_i J_{ij}$, so J is precisely the transition matrix between them. The connection potentials in the frames e_i and $\tilde{e}_i$ are $J^{-1}\tilde{\partial}J$ and $J^{-1}\partial J$ respectively, where we again have the same freedoms as before in P and $\tilde{P}$. To make further progress, we now specialise to the gauge where $A_w = A_z = 0$, corresponding to working in the frame e_i, and re-examine the ASD equations in this gauge.

$$\partial_{\tilde{z}}A_{\tilde{w}} - \partial_{\tilde{w}}A_{\tilde{z}} + [A_{\tilde{z}}, A_{\tilde{w}}] = 0, \quad \partial_z A_{\tilde{z}} - \partial_w A_{\tilde{w}} = 0. \tag{2.54}$$

Now if we take the second equation as an integrability condition, we get that there exists a matrix valued function K such that

$$A = \partial_z K\mathrm{d}\tilde{w} + \partial_w K\mathrm{d}\tilde{z}. \tag{2.55}$$

Like with the J matrix, K determines D and K is determined by D up to

$$K \mapsto g^{-1}Kg + zg^{-1}\partial_{\tilde{w}}g + wg^{-1}\partial_{\tilde{z}}g + c, \tag{2.56}$$

where g and c only depend on $\tilde{w}$ and $\tilde{z}$. As a condition on K, the ASD equation is

$$\partial_z\partial_{\tilde{z}}K - \partial_w\partial_{\tilde{w}}K + [\partial_w K, \partial_z K] = 0. \tag{2.57}$$

This can be easily seen by substituting (2.55) into (2.54).

2.2.5 Lagrangians for the ASDYM Equation

Both Yang's equation for the J matrix and Eq. (2.57) for the K matrix can be obtained from Lagrangians. We start with the K matrix case, which written in general linear coordinates is given by

$$\eta^{ab}\partial_a\partial_b K - 2\tilde{\alpha}^{ab}[\partial_a K, \partial_b K] = 0, \tag{2.58}$$

where $\tilde{\alpha}$ is defined in Eq. (2.25). This is the Euler Lagrange equation for the action

$$S[K] = \int \mathrm{tr}\left(\frac{1}{2}\eta^{ab}\partial_a K \partial_b K + \frac{4}{3}\tilde{\alpha}^{ab} K \partial_a K \partial_b K\right)\nu. \tag{2.59}$$

The J matrix is more subtle and does not have an invariant first order representation. Donaldson [6] observed that one can define a closed one-form Λ on the space of matrix-valued functions on space-time by

$$\Lambda(\delta J) = -\int \mathrm{tr}\left(\phi\partial\left(J^{-1}\tilde{\partial}J\right)\right)\wedge\omega, \quad \phi = J^{-1}\delta J, \tag{2.60}$$

where ω is again defined in Eq. (2.25) and $\mathrm{d} = \partial + \tilde{\partial}$. Stokes' theorem can be used here to discard boundary terms and prove closure, so we can then write

$$\Lambda(\delta J) = \delta S. \tag{2.61}$$

The variational principle $\delta S = 0$ is equivalent to Yang's equation in the form

$$\partial(J^{-1}\tilde{\partial}J)\wedge\omega = 0. \tag{2.62}$$

One form of the action from Donaldson is given by

$$S_\kappa = \frac{1}{2}\int \kappa\,\mathrm{tr}\,(F\wedge F)\,, \tag{2.63}$$

where F is the curvature 2-form of the connection $\mathrm{d} + J^{-1}\tilde{\partial}J$ and κ is any function such that $\omega = \partial\tilde{\partial}\kappa$. Another is given by the following. We write $J = UL^{-1}$, where L is strictly lower triangular and U is upper triangular with 1's along the diagonal. The action used to derive Yang's equation is then proposed to be [1]

$$S[J] = \frac{1}{2}\int \mathrm{tr}(2U^{-1}\tilde{\partial}U\wedge L^{-1}\partial L - L^{-1}\partial L\wedge L^{-1}\tilde{\partial}L)\wedge\omega. \tag{2.64}$$

Proof:
Define

$$\lambda = L^{-1}\partial L, \quad \tilde{\lambda} = L^{-1}\tilde{\partial}L, \quad \tilde{\mu} = U^{-1}\tilde{\partial}U. \tag{2.65}$$

In terms of these one-forms, the action is given by

$$S = \frac{1}{2}\int \mathrm{tr}\left(2\tilde{\mu}\wedge\lambda - \lambda\wedge\tilde{\lambda}\right)\wedge\omega, \tag{2.66}$$

and Yang's equation is

$$(\partial\tilde{\mu} + \tilde{\partial}\lambda + \tilde{\mu}\wedge\lambda + \lambda\wedge\tilde{\mu})\wedge\omega = 0. \tag{2.67}$$

Now we must show that $\delta S = 0$ coincides with Yang's equation. Setting $u = U^{-1}\delta U$ and $\ell = L^{-1}\delta L$, a short computation using $\delta(U^{-1}) = -U^{-1}(\delta U)U^{-1}$ gives

$$\delta\tilde{\mu} = \tilde{\partial}u + [\tilde{\mu}, u]. \tag{2.68}$$

Similarly, we have

$$\delta\lambda = \partial\ell + [\lambda, \ell], \quad \delta\tilde{\lambda} = \tilde{\partial}\ell + [\tilde{\lambda}, \ell]. \tag{2.69}$$

For $\delta L = 0$, the variation of the action becomes

$$\delta S = -\int_D \mathrm{tr}\left(u\left(\tilde{\partial}\lambda + \tilde{\mu}\wedge\lambda + \lambda\wedge\tilde{\mu}\right)\right)\wedge\omega + \int_{\partial D}\mathrm{tr}(u\lambda)\wedge\omega, \tag{2.70}$$

where D is the region of integration. Setting $\delta S = 0$ and discarding the boundary term, the first variation yields the bulk condition $\tilde{\partial}\lambda + \tilde{\mu}\wedge\lambda + \lambda\wedge\tilde{\mu} = 0$. For the second variation, we set $\delta U = 0$

and again perform integration by parts, using the fact that $\partial\tilde{\lambda} + \tilde{\partial}\lambda = -\lambda \wedge \tilde{\lambda} - \tilde{\lambda} \wedge \lambda$ is lower triangular and hence traceless to simplify the resulting expression:

$$\begin{aligned}\delta S &= \frac{1}{2}\int_D \operatorname{tr}\left(2\tilde{\mu} \wedge (\partial\ell + [\lambda, \ell]) - \partial\ell \wedge \tilde{\lambda} - \lambda \wedge \tilde{\partial}\ell\right) \wedge \omega \\ &= \int_D \operatorname{tr}\left(\ell(\partial\tilde{\mu} + \tilde{\mu} \wedge \lambda + \lambda \wedge \tilde{\mu} + \tilde{\partial}\lambda)\right) \wedge \omega + \int_{\partial D} \operatorname{tr}\left(-\ell\tilde{\mu} + \frac{1}{2}\ell(\tilde{\lambda} - \lambda)\right) \wedge \omega.\end{aligned} \tag{2.71}$$

Setting both variations to zero and discarding boundary terms, the bulk conditions from each variation combine to give Yang's equation. □

These actions are closely related to the four-dimensional Wess–Zumino–Witten model that will arise naturally from holomorphic Chern–Simons theory on twistor space in Section 3.2.

2.2.6 The Hamiltonian Formalism

Having established Lagrangian formulations of the ASDYM equation via the J and K matrices, the Hamiltonian picture is developed next. The goal is to show that the two formulations are equivalent not just at the level of equations of motion, but also at the level of the symplectic geometry of the solution space, i.e., they determine the same symplectic structure. Given a Lagrangian field theory, the boundary terms arising in the variational calculation naturally define a symplectic potential Θ, a one-form on the solution space whose exterior derivative $\Omega = \mathrm{d}\Theta$ gives the symplectic structure. For a translation-invariant Lagrangian density L depending on fields ξ_i and their derivatives $\xi_{ai} = \partial_a \xi_i$, this takes the form

$$\Theta(\delta\xi_i) = \int \frac{\partial L}{\partial \xi_{ai}} \delta\xi_i \Sigma_a, \tag{2.72}$$

where $\delta\xi_i$ is a solution to the linearized field equations, $\Sigma_a = \partial_a \lrcorner \nu$, and the integral is taken over a Cauchy surface. When L is independent of the coordinates x^a, the translations in space-time determine Hamiltonian flows on the solution space, with translation along the constant vector field $V = V^a \partial_a$ generated by the Hamiltonian

$$H_V = \int V^b \left(\frac{\partial L}{\partial \xi_{ai}} \xi_{bi} - \delta^a_b L\right) \Sigma_a. \tag{2.73}$$

To make sense of Hamiltonian flows on the solution space, we must first understand the geometry of that space itself. The "solution space" of the ASDYM equation is the quotient $\mathcal{M} = \mathcal{C}/\mathcal{G}$, where $\mathcal{C}$ is the set of ASD connections on a fixed vector bundle $E \to U$, and $\mathcal{G}$ is the group of active gauge transformations. Two connections $D, D' \in \mathcal{C}$ determine the same point in $\mathcal{M}$ if there exists an automorphism such that $g(E_x) = E_x$ for all $x \in U$, and

$$D(gs) = gD's \tag{2.74}$$

for every section s. There is a projection that maps $\mathcal{C} \to \mathcal{M}$ by mapping a connection D to its equivalence class $[D]$. If $\mathcal{J}$ and $\mathcal{K}$ represent the solution spaces for the J and K matrix formulations of the ASD equation, then there exist projections

$$\mathcal{J} \to \mathcal{M}, \quad \mathcal{K} \to \mathcal{M}, \tag{2.75}$$

which map the J and K matrices to the corresponding solutions to the ASDYM equation. To identify tangent vectors to $\mathcal{C}$, the linearized curvature is computed. If we take a general connection $D = \mathrm{d} + A$ on E with curvature F, one can obtain the curvature of $D + \psi$ to first order in ψ by the following:

$$\begin{aligned}F' &= \mathrm{d}(A + \psi) + 2(A + \psi) \wedge (A + \psi) \\ &= \underbrace{\mathrm{d}A + A \wedge A}_{=F} + \underbrace{\mathrm{d}\psi + A \wedge \psi + \psi \wedge A}_{=D\psi} + \underbrace{\psi \wedge \psi}_{=\mathcal{O}(\psi^2)} \\ &= F + D\psi.\end{aligned} \tag{2.76}$$

Here ψ is a one-form with values in $\operatorname{adj}(E)$. Since F is anti-self-dual, the linearized ASDYM equation is

$$D\psi = - * D\psi. \tag{2.77}$$

A solution to this represents a tangent vector to $\mathcal{C}$ at D and hence, by projection, an element of $T_{[D]}\mathcal{M}$. The projection vanishes whenever ψ is an infinitesimal gauge transformation of the connection, i.e., whenever it takes the form

$$\psi = \mathrm{d}\phi + [A, \phi] = D\phi \tag{2.78}$$

for some section ϕ of the adjoint bundle. Hence a tangent to $\mathcal{M}$ is represented by a class of gauge-equivalent (differ by some $D\phi$) one-forms ψ, and we can choose a ϕ such that

$$\psi = \psi_{\tilde{w}}\mathrm{d}\tilde{w} + \psi_{\tilde{z}}\mathrm{d}\tilde{z}, \tag{2.79}$$

which is unique up the addition of a $D\phi$ where $D_w\phi = D_z\phi = 0$. As discussed before, an ASDYM potential can be written in two equivalent forms:

$$A = J^{-1}\tilde{\partial}J = \partial_z K \mathrm{d}\tilde{w} + \partial_w K \mathrm{d}\tilde{z}. \tag{2.80}$$

If we consider $\delta J = J\phi$ and $\delta K = \phi$, where ϕ here is a matrix valued function on space-time not to be confused with the gauge parameter appearing earlier, then δJ and δK satisfy the respective linearized equations if and only if

$$\partial_w \partial_{\tilde{w}}\phi - \partial_z \partial_{\tilde{z}}\phi + [A_{\tilde{w}}, \partial_w \phi] - [A_{\tilde{z}}, \partial_z \phi] = 0. \tag{2.81}$$

To show this, we start by applying the proposed variation to J and ensuring that it satisfies Yang's equation.

$$\begin{aligned} J^{-1}\partial_{\tilde{w}}J &\to (J(1+\phi))^{-1}\,\partial_{\tilde{w}}\left(J(1+\phi)\right) \\ &= (1+\phi)^{-1}J^{-1}\partial_{\tilde{w}}\left(J(1+\phi)\right) \\ &\approxeq (1-\phi)J^{-1}\partial_{\tilde{w}}\left(J(1+\phi)\right) \\ &= (1-\phi)J^{-1}\left((\partial_{\tilde{w}}J)(1+\phi) + J\partial_{\tilde{w}}\phi\right) \\ &\approxeq J^{-1}\partial_{\tilde{w}}J + \partial_{\tilde{w}}\phi + [J^{-1}\partial_{\tilde{w}}J, \phi]. \end{aligned} \tag{2.82}$$

Similarly,

$$J^{-1}\partial_{\tilde{z}}J \to J^{-1}\partial_{\tilde{z}}J + \partial_{\tilde{z}}\phi + [J^{-1}\partial_{\tilde{z}}J, \phi]. \tag{2.83}$$

Now we substitute these into Eq. (2.50), giving

$$\begin{aligned} &\partial_w\left(J^{-1}\partial_{\tilde{w}}J + \partial_{\tilde{w}}\phi + [J^{-1}\partial_{\tilde{w}}J, \phi]\right) - \partial_z\left(J^{-1}\partial_{\tilde{z}}J + \partial_{\tilde{z}}\phi + [J^{-1}\partial_{\tilde{z}}J, \phi]\right) = 0 \\ \implies &\partial_w\left(\partial_{\tilde{w}}\phi + [J^{-1}\partial_{\tilde{w}}J, \phi]\right) - \partial_z\left(\partial_{\tilde{z}}\phi + [J^{-1}\partial_{\tilde{z}}J, \phi]\right) = 0 \\ \implies &\partial_w\partial_{\tilde{w}}\phi - \partial_z\partial_{\tilde{z}}\phi + [\underbrace{J^{-1}\partial_{\tilde{w}}J}_{=A_{\tilde{w}}}, \partial_w\phi] - [\underbrace{J^{-1}\partial_{\tilde{z}}J}_{=A_{\tilde{z}}}, \partial_z\phi] = 0. \end{aligned} \tag{2.84}$$

Now we must check the condition on the variation of K by substituting it into Eq. (2.57), again going to linear order.

$$\begin{aligned} &K \to K + \phi \\ \implies &\partial_z\partial_{\tilde{z}}(K+\phi) - \partial_w\partial_{\tilde{w}}(K+\phi) + [\partial_w(K+\phi), \partial_z(K+\phi)] = 0 \\ \implies &\partial_z\partial_{\tilde{z}}\phi - \partial_w\partial_{\tilde{w}}\phi + [\partial_w K, \partial_z\phi] + [\partial_w\phi, \partial_z K] + \mathcal{O}(\phi^2) = 0 \\ \implies &\partial_w\partial_{\tilde{w}}\phi - \partial_z\partial_{\tilde{z}}\phi + [\underbrace{\partial_z K}_{=A_{\tilde{w}}}, \partial_w\phi] - [\underbrace{\partial_w K}_{=A_{\tilde{z}}}, \partial_z\phi] = 0 \end{aligned} \tag{2.85}$$

Hence, the condition on ϕ is the same for both the J and K variations. Under gauge transformations, ϕ behaves as a section of the adjoint bundle and Eq. (2.81) is the background-coupled wave equation

$$D * D\phi = 0. \tag{2.86}$$

D is the connection $\mathrm{d} + [A, \cdot]$ on the adjoint bundle and $*$ is the Hodge operator on 1-forms, which maps 1 forms to 3-forms.

$$*\beta = (\beta_w \mathrm{d}w + \beta_z \mathrm{d}z - \beta_{\tilde{w}}\mathrm{d}\tilde{w} - \beta_{\tilde{z}}\mathrm{d}\tilde{z}) \wedge \omega, \text{ where } \omega = \mathrm{d}w \wedge \mathrm{d}\tilde{w} + \mathrm{d}z \wedge \mathrm{d}\tilde{z}. \tag{2.87}$$

We write D as the sum of a holomorphic and an anti-holomorphic operator $D = \partial_D + \tilde{\partial}_D$, such that

$$\partial_D = \mathrm{d}wD_w + \mathrm{d}zD_z, \quad \tilde{\partial}_D = \mathrm{d}\tilde{w}D_{\tilde{w}} + \mathrm{d}\tilde{z}D_{\tilde{z}}. \tag{2.88}$$

Denoting the solution space to the background coupled wave equation of an ASD connection D as $\mathcal{W}_D$, we have two subspaces $\mathcal{W}_{D+}$ and $\mathcal{W}_{D-}$ corresponding to $\tilde{\partial}_D\phi = 0$ and $\partial_D\phi = 0$ respectively. Writing δJ and δK and constructing D from J or K as above, we have

$$T_J\mathcal{J} = \mathcal{W}_D = T_K\mathcal{K}, \tag{2.89}$$

as well as the projection maps

$$T_J\mathcal{J} \to T_{[D]}\mathcal{M}, \quad T_K\mathcal{K} \to T_{[D]}\mathcal{M}, \tag{2.90}$$

which are defined by mapping δJ and δK to their corresponding solution to the linearized ASDYM equation, given by

$$\psi = D_{\tilde{w}}\phi\,\mathrm{d}\tilde{w} + D_{\tilde{z}}\phi\,\mathrm{d}\tilde{z} \quad \text{and} \quad \psi' = D_z\phi\,\mathrm{d}\tilde{w} + D_w\phi\,\mathrm{d}\tilde{z}. \tag{2.91}$$

If $\phi \in \mathcal{W}_{D+}$, then $\psi = 0$ and if $\phi \in \mathcal{W}_{D-}$, then $\psi' = 0$. The Lagrangian densities for equations (2.50) and (2.57) determine symplectic structures on $\mathcal{J}$ and $\mathcal{K}$, and as a result determine two bilinear forms on $\mathcal{W}_D$ for each solution to the ASDYM equation. These bilinear forms coincide, and are given by

$$\begin{aligned}\Omega(\phi,\phi') &= \frac{1}{2}\int \mathrm{tr}(\phi * D\phi' - \phi' * D\phi) \\ &= \frac{1}{2}\int \mathrm{tr}\left(\phi\partial_D\phi' - \phi\tilde{\partial}_D\phi' - \phi'\partial_D\phi + \phi'\tilde{\partial}_D\phi\right)\wedge\omega.\end{aligned} \tag{2.92}$$

We notice they only depend on D, and are the same if J or K is replaced by another potential for the same connection. For the K-matrix, this follows from writing $A^a = 2\tilde{\alpha}^{ab}\partial_b K$. It is less obvious for the J matrix, we must show that the exterior derivative of the 1 form

$$\Theta(\delta J) = \int \mathrm{tr}\left(u\lambda - \ell\tilde{\mu} - \frac{1}{2}\ell(\lambda - \tilde{\lambda})\right)\wedge\omega \tag{2.93}$$

coincides at each solution on $\mathcal{J}$ with Ω. To show this, we define two commuting variations δ and δ', i.e. two commuting vector fields on the solution space of Yang's equation, and define ϕ, u and ℓ as before. We set $\tilde{\phi} = \delta J J^{-1}$, with primes denoting the change of $\delta \to \delta'$. This gives us

$$\phi = L(u-\ell)L^{-1}, \quad \tilde{\phi} = U(u-\ell)U^{-1}, \tag{2.94}$$

as well as

$$\mathrm{tr}\left(\phi\partial\phi' - \tilde{\phi}\tilde{\partial}\tilde{\phi}'\right) = \mathrm{tr}\left((u-\ell)\partial(u'-\ell') + (u-\ell)\tilde{\partial}(u'-\ell') + (u-\ell)[\lambda-\tilde{\mu}, u'-\ell']\right). \tag{2.95}$$

We also have

$$\mathrm{tr}\left(\delta(u'\lambda) - \delta'(u\lambda)\right) = \mathrm{tr}\left(u'(\partial\ell + [\lambda,\ell]) - u(\partial\ell' + [\lambda,\ell']) - [u,u']\lambda\right), \tag{2.96}$$

and analogous identities hold for $\delta\lambda$, $\delta\tilde{\lambda}$. By using Stokes' theorem to kill exact differential terms, and noticing that the trace of a strictly upper triangular or lower triangular matrix is obviously zero, we can say that

$$\delta(\Theta(\delta' J)) - \delta'(\Theta(\delta J)) = \frac{1}{2}\int \mathrm{tr}\left(\phi\partial\phi' - \phi'\partial\phi + \tilde{\phi}\tilde{\partial}\tilde{\phi}' - \tilde{\phi}'\tilde{\partial}\tilde{\phi}\right)\wedge\omega. \tag{2.97}$$

Now $\partial_D\phi = \partial\phi$ and

$$\tilde{\partial}_D\phi = \tilde{\partial}\phi + [J^{-1}\tilde{\partial}J, \phi] = J^{-1}(\tilde{\partial}\tilde{\phi})J. \tag{2.98}$$

All of this together gives that $\Omega(\phi,\phi')$ is equal to the right-hand side of (2.97), and $\mathrm{d}\Theta = \Omega$. For the K-matrix formulation, the Hamiltonian generating translations along V is

$$H_V = \int \mathrm{tr}V^b\left(\partial^a K\partial_b K - \frac{1}{2}\delta^a_b\partial^c K\partial_c K + \frac{4}{3}\tilde{\alpha}^{ac}K[\partial_b K, \partial_c K] - \frac{4}{3}\delta^a_b\tilde{\alpha}^{cd}K\partial_c K\partial_d K\right)\Sigma_a. \tag{2.99}$$

This establishes that $d\Theta = \Omega$ for the J-matrix formulation, confirming that both the J and K matrices determine the same symplectic structure on the solution space of the ASDYM equation.

2.3 Chern–Simons theory

Chern–Simons theory is a topological quantum field theory. It was first developed by Chern and Simons in 1974 [7] and has subsequently played an important role in mathematical physics, for example in the work of Witten [8].

2.3.1 Three-dimensional Chern–Simons theory

Action and gauge invariance

The Chern–Simons action and three-form are defined by the following:

$$S_{3\mathrm{dCS}}[A] = \frac{k}{4\pi}\int_{M_3} \mathrm{CS}[A], \quad \mathrm{CS}[A] = \mathrm{tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right). \tag{2.100}$$

It is not strictly gauge-invariant, but its variation under a gauge transformation differs by quantised terms.

$$\begin{aligned} A \to A^g &= g^{-1}Ag + g^{-1}\mathrm{d}g, \\ \mathrm{CS}[A] \to \mathrm{CS}[A^g] &= \mathrm{tr}\left(A^g\wedge \mathrm{d}A^g + \frac{2}{3}A^g\wedge A^g\wedge A^g\right). \end{aligned} \tag{2.101}$$

For convenience, define

$$\alpha := g^{-1}Ag, \qquad \beta := g^{-1}\mathrm{d}g, \tag{2.102}$$

so that

$$A^g = \alpha + \beta. \tag{2.103}$$

We first compute the differentials:

$$\begin{aligned} \mathrm{d}\alpha &= \mathrm{d}(g^{-1}Ag) = \mathrm{d}(g^{-1})\wedge Ag + g^{-1}\mathrm{d}A\, g - g^{-1}A\wedge \mathrm{d}g \\ &= -g^{-1}\mathrm{d}g\, g^{-1}\wedge Ag + g^{-1}\mathrm{d}A\, g - g^{-1}A\wedge \mathrm{d}g \\ &= -\beta\wedge\alpha + g^{-1}\mathrm{d}A\, g - \alpha\wedge\beta, \\ \mathrm{d}\beta &= \mathrm{d}(g^{-1}\mathrm{d}g) = \mathrm{d}(g^{-1})\wedge \mathrm{d}g + g^{-1}\mathrm{d}^2 g = -g^{-1}\mathrm{d}g\, g^{-1}\wedge \mathrm{d}g = -\beta\wedge\beta, \end{aligned} \tag{2.104}$$

where we used $\mathrm{d}(g^{-1}) = -g^{-1}\mathrm{d}g\, g^{-1}$ and $\mathrm{d}^2 g = 0$. Thus,

$$\mathrm{d}A^g = \mathrm{d}\alpha + \mathrm{d}\beta = g^{-1}\mathrm{d}A\, g - (\alpha\wedge\beta + \beta\wedge\alpha) - \beta\wedge\beta. \tag{2.105}$$

Using cyclicity of the trace and the fact that a cyclic permutation of three 1-forms introduces no sign,

$$\mathrm{tr}(X\wedge Y\wedge Z) = \mathrm{tr}(Y\wedge Z\wedge X) = \mathrm{tr}(Z\wedge X\wedge Y), \tag{2.106}$$

the cubic term simplifies to

$$\mathrm{tr}(\alpha+\beta)^3 = \mathrm{tr}(\alpha^3) + 3\,\mathrm{tr}(\alpha^2\wedge\beta) + 3\,\mathrm{tr}(\alpha\wedge\beta^2) + \mathrm{tr}(\beta^3). \tag{2.107}$$

Substituting $\mathrm{d}A^g$ and expanding $\mathrm{CS}[A^g]$, collecting terms and using $\mathrm{tr}(\alpha\wedge g^{-1}\mathrm{d}A\, g) = \mathrm{tr}(A\wedge \mathrm{d}A)$ and $\mathrm{tr}(\alpha^3) = \mathrm{tr}(A^3)$ to reconstruct $\mathrm{CS}[A]$, one finds

$$\mathrm{CS}[A^g] = \mathrm{CS}[A] + \mathrm{tr}\left(\beta\wedge g^{-1}\mathrm{d}A\, g - \alpha\wedge\beta\wedge\beta - \frac{1}{3}\beta\wedge\beta\wedge\beta\right). \tag{2.108}$$

The terms purely in β combine to

$$-\frac{1}{3}\,\mathrm{tr}\big(\beta\wedge\beta\wedge\beta\big) = -\frac{1}{3}\,\mathrm{tr}\big(g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\big). \tag{2.109}$$

All mixed terms in α and β can be regrouped into an exact differential. The key identity we want to justify is that the mixed terms in $\mathrm{CS}[A^g]$ (those containing both α and β) can be written as

$$\mathrm{d}\ \mathrm{tr}(\alpha\wedge\beta) = \mathrm{d}\ \mathrm{tr}(A\wedge \mathrm{d}g\, g^{-1}). \tag{2.110}$$

We start by computing $\mathrm{d}\,\mathrm{tr}(\alpha\wedge\beta)$ explicitly. Since α and β are 1-forms, we have

$$\mathrm{d}(\alpha\wedge\beta)=\mathrm{d}\alpha\wedge\beta-\alpha\wedge\mathrm{d}\beta, \tag{2.111}$$

hence,

$$\mathrm{d}\,\mathrm{tr}(\alpha\wedge\beta)=\mathrm{tr}\big(\mathrm{d}\alpha\wedge\beta-\alpha\wedge\mathrm{d}\beta\big). \tag{2.112}$$

Substituting earlier expressions for $\mathrm{d}\alpha$ and $\mathrm{d}\beta$, we get:

$$\begin{aligned}\mathrm{d}\,\mathrm{tr}(\alpha\wedge\beta)&=\mathrm{tr}\big((g^{-1}\mathrm{d}A\,g-(\alpha\wedge\beta+\beta\wedge\alpha))\wedge\beta-\alpha\wedge(-\beta\wedge\beta)\big)\\&=\mathrm{tr}\big(g^{-1}\mathrm{d}A\,g\wedge\beta-(\alpha\wedge\beta+\beta\wedge\alpha)\wedge\beta+\alpha\wedge\beta\wedge\beta\big)\\&=\mathrm{tr}\big(g^{-1}\mathrm{d}A\,g\wedge\beta-\alpha\wedge\beta\wedge\beta\big).\end{aligned} \tag{2.113}$$

Collecting everything,

$$\mathrm{CS}[A^g]=\mathrm{CS}[A]+\mathrm{d}\,\mathrm{tr}\big(A\wedge\mathrm{d}gg^{-1}\big)-\frac{1}{3}\,\mathrm{tr}\big(g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\big). \tag{2.114}$$

In the case where M_3 is a manifold without boundary, the total derivative term does not contribute to the action. The third term seems like an issue but actually is not, as it is quantised,

$$\frac{1}{3}\int_{M_3}\mathrm{tr}\big(g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\big)\in 8\pi^2\mathbb{Z}. \tag{2.115}$$

The proof for this is in Appendix A. This will not contribute to the path integral as long as the level k is also quantised:

$$\exp(iS_{3\mathrm{dCS}}[A^g])=\exp(iS_{3\mathrm{dCS}}[A])\quad\text{for } k\in\mathbb{Z}. \tag{2.116}$$

The equations of motion for 3dCS theory are given by

$$F[A]=\mathrm{d}A+A\wedge A=0. \tag{2.117}$$

This theory therefore describes flat gauge fields, which locally can be written as pure gauge, i.e., $A=g^{-1}\mathrm{d}g$. As a result, there are no local dynamics in this theory, as A can always be set to zero by a gauge transformation. While this may appear to have trivial dynamics, some interesting topological or non-local features can be observed in this theory. Contrary to standard quantum field theories, the natural observables are not local operators, but rather are Wilson lines defined over a 1d curve $\gamma\in M_3$,

$$W(\gamma)=P\exp\int_\gamma A. \tag{2.118}$$

For flat gauge fields these line operators are topological, which means they are invariant under smooth deformations of the curve γ. Witten's original paper [8] related correlation functions of these Wilson lines to the Jones polynomial.

Boundaries and Edge modes

We wish to consider this theory in the context where $M^3=\Sigma\times\mathbb{R}_{\geq 0}$ with radial direction $r\in\mathbb{R}_{\geq 0}$, boundary 2-manifold Σ with flat Lorentzian metric $\mathrm{d}s^2=\mathrm{d}t^2-\mathrm{d}x^2$ (light-cone $\partial_\pm=\partial_t\pm\partial_x$). Writing the action out in components [9], we have:

$$\begin{aligned}S_{3\mathrm{dCS}}[A]=&\frac{k}{4\pi}\int_{M^3}\mathrm{d}^3x\;\mathrm{tr}\left(A_t\left(\partial_xA_r-\partial_rA_x\right)+A_x\left(\partial_rA_t-\partial_tA_r\right)+A_r\left(\partial_tA_x-\partial_xA_t\right)+\right.\\&\left.+2A_t(A_xA_r-A_rA_x)\right)\\=&\frac{k}{4\pi}\int_{M^3}\mathrm{d}^3x\;\mathrm{tr}\left(2A_t\underbrace{\left(\partial_xA_r-\partial_rA_x+[A_x,A_r]\right)}_{=F_{xr}}+A_x\left(\partial_rA_t-\partial_tA_r\right)+\right.\\&\left.A_r\left(\partial_tA_x-\partial_xA_t\right)-A_t\partial_xA_r+A_t\partial_rA_x\right).\end{aligned} \tag{2.119}$$

Using integration by parts, we have

$$S_{3\mathrm{dCS}}[A] = \frac{k}{4\pi}\int_{M^3} \mathrm{d}^3x\ \mathrm{tr}\left(A_tF_{xr} + A_r\partial_tA_x - A_x\partial_tA_r\right) + \frac{k}{4\pi}\int_\Sigma \mathrm{d}^2x\ \mathrm{tr}(A_xA_t). \tag{2.120}$$

Bulk terms show A_t as a Lagrange multiplier enforcing $F_{xr} = 0$, with conjugate momenta $P_x = A_r$, $P_r = -A_x$ (up to normalization). This means that A_x and A_r form a complete set of phase space coordinates on a fixed time slice. Due to this, we can interpret the boundary variation of the action as a symplectic potential on phase space,

$$\theta = \int_\Sigma \mathrm{tr}(\delta A \wedge A). \tag{2.121}$$

Boundary conditions here are set entirely by fixing the field configuration on the boundary (Dirichlet boundary conditions). There is no way to impose any von Neumann boundary conditions here, as the derivative of the field is not a phase space coordinate. This is a key difference between first- and second-order systems. One choice of boundary conditions is given by

$$A_+|_\Sigma = 0 \implies \delta A_+|_\Sigma = 0. \tag{2.122}$$

Applying a gauge transformation to the action, the boundary term vanishes if we impose that

$$\partial_+ g|_\Sigma = 0. \tag{2.123}$$

This can be seen by taking a gauge transformation $A^g = g^{-1}Ag + g^{-1}\mathrm{d}g$ and imposing the boundary conditions again $A^g_+|_\Sigma = 0$. Imposing $\partial_+ g|_\Sigma = 0$ ensures that this is indeed the case.

One might be concerned that by introducing a metric on the boundary, we have affected the topological nature of the theory. Earlier it was mentioned that A can be written locally as $g^{-1}\mathrm{d}g$, and hence we can always set $A = 0$ by a gauge transformation. However, now there is the additional point that g must satisfy the boundary conditions imposed, so it may no longer be possible to do this. In some cases, degrees of freedom of A survive gauge fixing and generate local dynamics on the boundaries. These are called edge modes. Let's try to see this for the action obtained earlier. We have from this that $F_{xr} = 0$ and $A_x|_\Sigma = -A_t|_\Sigma$, so we can integrate out the first term and replace A_t with $-A_x$ on the boundary. Setting the gauge potential to the Maurer-Cartan form $A = j = g^{-1}\mathrm{d}g$, we have

$$S[g] = \frac{k}{4\pi}\int_{M^3} \mathrm{d}^3x\,\mathrm{tr}(j_r\partial_t j_x - j_x\partial_t j_r) - \frac{k}{4\pi}\int_{M^3} \mathrm{d}^2x\,\mathrm{tr}(j_xj_x). \tag{2.124}$$

Employing the Maurer-Cartan equation $\mathrm{d}j + j\wedge j = 0$ along with an integration by parts gives us

$$S_{\mathrm{cWZW}}[g] = -\frac{k}{4\pi}\int_\Sigma \mathrm{d}^2x\,\mathrm{tr}(j_xj_+) - \frac{k}{12\pi}\int_{M^3} \mathrm{tr}(j\wedge j\wedge j). \tag{2.125}$$

This is the "chiral" Wess–Zumino–Witten (WZW) action, which has an unusual kinetic term. This theory only encodes half of the usual degrees of freedom in a WZW model. The full WZW model can be recovered by considering $M_3 = \Sigma\times[0,1]$ and imposing the boundary conditions

$$A_+|_{\Sigma\times\{0\}} = 0, \quad A_-|_{\Sigma\times\{1\}} = 0. \tag{2.126}$$

This leads to two chiral WZW models, one on each boundary. The residual symmetries preserving these boundary conditions provide the chiral and anti-chiral symmetries of the full WZW model.

2.3.2 Four-dimensional Chern–Simons theory

The aim of extending Chern–Simons to a four-dimensional theory is to be able to work on a four-manifold $M_4 = C\times\Sigma$, where C is a complex curve. The aim in doing this is that C will be used to parametrise the complex spectral parameter $\zeta\in\mathbb{C}$ for the Lax formalism of lower dimensional integrable models. Choices of this complex curve C include $\mathbb{CP}^1 = \mathbb{C}\times\{\infty\}$, $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $T^2 = S^1\times S^1$, which correspond to rational, trigonometric, and elliptic integrable models respectively.

We will mainly work with $\mathbb{CP}^1$. This 4d extension of Chern–Simons was introduced in [10] and further studied in [11, 2]. To start, we must introduce a one-form ω to define a new action:

$$S_{4\text{dCS}}[A] = \frac{1}{2\pi i}\int_{M_4} \omega \wedge \text{CS}[A]. \tag{2.127}$$

The sufficient and necessary constraints on ω are currently not known, so we will assume it is the pullback of a general meromorphic (1,0)-form. Examples of such choices for ω are

$$\mathrm{d}\zeta, \quad \frac{\mathrm{d}\zeta}{\zeta}, \quad \frac{1-\zeta^2}{\zeta^2}\mathrm{d}\zeta. \tag{2.128}$$

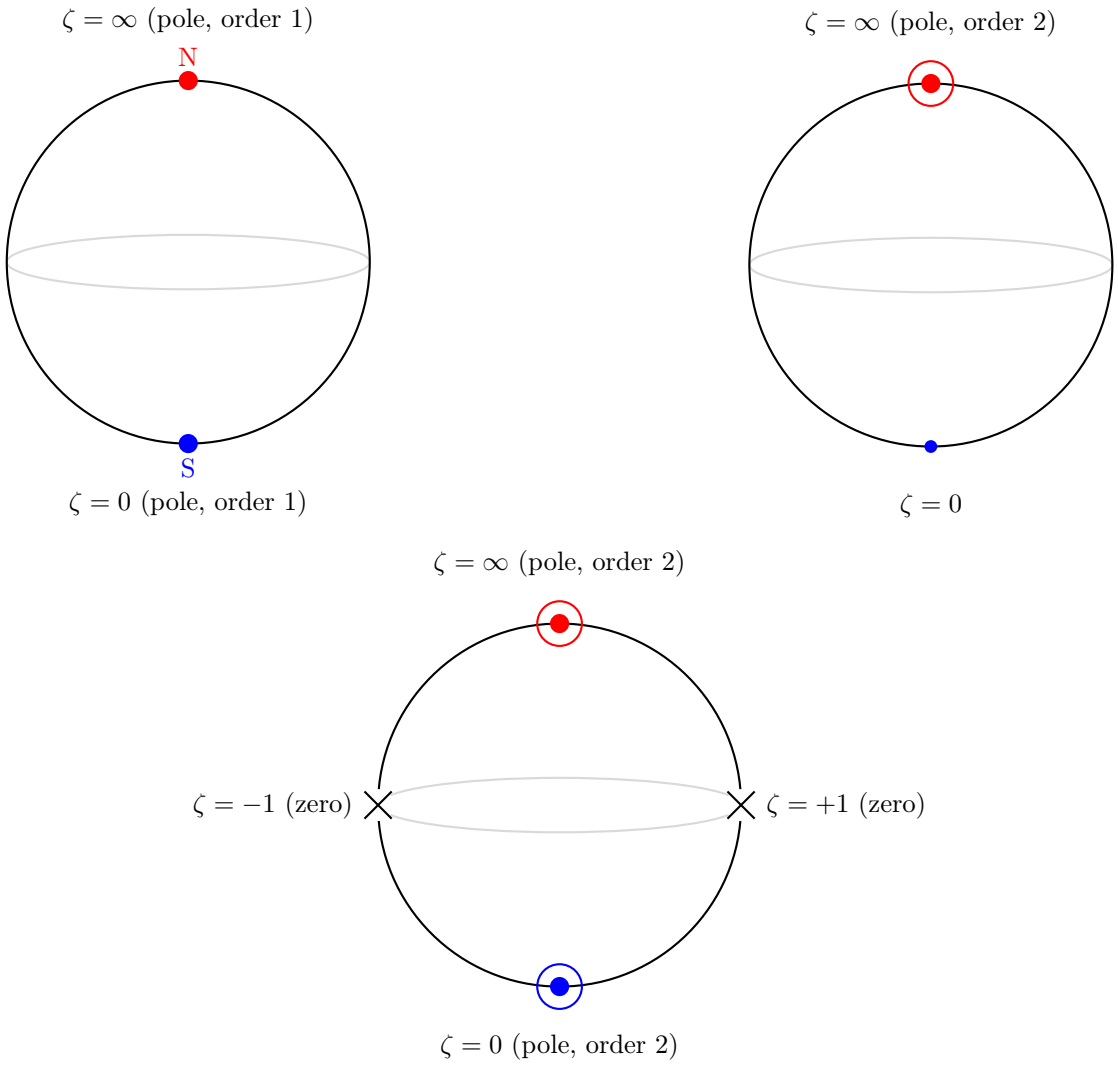


Figure 2.1: Poles and zeros of meromorphic 1-forms on $\mathbb{CP}^1$, illustrated on the Riemann sphere. *Left:* $\omega = \mathrm{d}\zeta$ has a double pole at $\zeta = \infty$ (North) and no pole or zero at $\zeta = 0$ (South). *Right:* $\omega = \frac{\mathrm{d}\zeta}{\zeta}$ has simple poles at both $\zeta = 0$ (South) and $\zeta = \infty$ (North). *Bottom:* $\omega = \frac{(1-\zeta^2)}{\zeta^2}\,\mathrm{d}\zeta$ has double poles at $\zeta = 0$ and $\zeta = \infty$, with simple zeros at $\zeta = \pm 1$.

These each have different properties, such as number of and location of poles/vanishing points, which can be used to describe different integrable models. The significance of the poles of ω can be understood by varying the action:

$$\delta S_{4\text{dCS}}[A] = \frac{1}{\pi i}\int_{M_4} \omega \wedge \operatorname{tr}(\delta A \wedge F) + \frac{1}{2\pi i}\int_{M_4} \mathrm{d}\omega \wedge \operatorname{tr}(\delta A \wedge A). \tag{2.129}$$

The first term gives the bulk equations of motion $\omega \wedge F = 0$, while the second term can be simplified by choosing an ω. We take $\omega = \frac{\mathrm{d}\zeta}{\zeta}$ as an example. One may think that $\mathrm{d}\omega = 0$ at first glance, but $\frac{1}{\zeta}$ is not a holomorphic function. We have

$$\partial_{\bar{\zeta}}\left(\frac{1}{\zeta}\right) = -2\pi i\,\delta(\zeta), \quad \frac{1}{2\pi i}\int_{\mathbb{CP}^1} \mathrm{d}\left(\frac{\mathrm{d}\zeta}{\zeta}\right) f(\zeta) = f(0) - f(\infty). \tag{2.130}$$

These two contributions can be seen from considering two different patches, with ζ and $\tilde{\zeta} = \frac{1}{\zeta}$ as the inhomogeneous coordinates on these patches. This will give us two terms corresponding to the two separate poles of our chosen ω at $\zeta = 0$ and $\zeta = \infty$, which can be seen in Fig. 2.1. The second term of the action can therefore be written as

$$\frac{1}{2\pi i}\int_{\mathbb{CP}^1\times\Sigma} \mathrm{d}\left(\frac{\mathrm{d}\zeta}{\zeta}\right) \wedge \operatorname{tr}(\delta A \wedge A) = \int_{\Sigma} \left(\operatorname{tr}(\delta A \wedge A)|_{\zeta=0} - \operatorname{tr}(\delta A \wedge A)|_{\zeta=\infty}\right). \tag{2.131}$$

This is the boundary term of this theory, and we must impose boundary conditions on A at $\zeta = 0$ and $\zeta = \infty$ to ensure that it vanishes.

Wess–Zumino–Witten model

The derivation in this section closely follows Cole [12]. Still using $\omega = \frac{\mathrm{d}\zeta}{\zeta}$, we can choose boundary conditions $A_-|_{\zeta=\infty} = 0$ and $A_+|_{\zeta=0} = 0$. Like in the 3d case, this restricts the gauge transformations to those which obey $\partial_+ g|_{\zeta=0} = 0$ and $\partial_- g|_{\zeta=\infty} = 0$. It is also important to note that transformations that act non-trivially at the poles of ω are not necessarily gauge transformations. As in the three-dimensional case, imposing boundary conditions breaks part of the gauge symmetry, producing edge modes that carry the dynamical content of the boundary theory. To isolate these, we decompose the bulk gauge field by introducing a new dynamical field $\hat{g}$ via:

$$A = L^{\hat{g}} \equiv \hat{g}^{-1}L\hat{g} + \hat{g}^{-1}\mathrm{d}\hat{g}. \tag{2.132}$$

While this has the same form as a gauge transformation, we are treating it as a new dynamical field for this theory. In particular, $\hat{g}$ is not subject to the same boundary constraints as before. The L here becomes a Lax connection for the boundary theory. The new action is given by

$$S_{4\text{dCS}}[L,\hat{g}] = \frac{1}{2\pi i}\int_{M_4} \left(\omega \wedge \mathrm{CS}[L] + \mathrm{d}\omega \wedge \operatorname{tr}(L \wedge \mathrm{d}\hat{g}\hat{g}^{-1}) + \omega \wedge \mathrm{CS}[\hat{g}^{-1}\mathrm{d}\hat{g}]\right). \tag{2.133}$$

The final term in this expression is known as the Wess–Zumino (WZ) 3-form:

$$\mathrm{CS}[\hat{g}^{-1}\mathrm{d}\hat{g}] = -\frac{1}{3}\operatorname{tr}\left(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}\right) \equiv -\mathrm{WZ}[\hat{g}]. \tag{2.134}$$

This is the same Wess–Zumino term encountered in the derivation of the chiral WZW model from three-dimensional Chern–Simons theory in Section 2.3.1, and its topological quantization properties are established in Appendix A. This also appears in the context of 2d WZW models, where it only generates boundary dynamics. It does something similar here, which can be shown by considering the extension $\hat{g} : M_5 \to G$, where $M_5 = M_4 \times [0,1]$. The requirement is that this is a smooth homotopy from the trivial map at $M_4 \times \{0\}$ to the edge mode $\hat{g}$ at $M_4 \times \{1\}$ such that the third term in the action can be written as

$$\int_{M_4} \omega \wedge \mathrm{WZ}[\hat{g}] = \int_{M_5} \mathrm{d}\left(\omega \wedge \mathrm{WZ}[\hat{g}]\right). \tag{2.135}$$

Since $\mathrm{d}\,\mathrm{WZ}[\hat{g}] = 0$, the exterior derivative only acts on ω in this expression. This localizes the integral to the poles of ω, giving us

$$S_{4\text{dCS}}[L,\hat{g}] = \frac{1}{2\pi i}\int_{M_4} \omega \wedge \mathrm{CS}[L] + \frac{1}{2\pi i}\int_{M_4} \mathrm{d}\omega \wedge \left(\operatorname{tr}(L \wedge \mathrm{d}gg^{-1}) - \int_{[0,1]} \mathrm{WZ}[\hat{g}]\right). \tag{2.136}$$

Here, we can see the action is only dependent on $\hat{g}$ through its values around the poles of ω, hence the title "edge mode". Now we wish to count the physical degrees of freedom and the symmetries of this theory. We can denote the degrees of freedom of $\hat{g}$ by

$$\hat{g}|_{\zeta=0} = g, \quad \hat{g}|_{\zeta=\infty} = \tilde{g}. \tag{2.137}$$

The symmetries of this theory fall into two distinct classes. Symmetries inherited from the original gauge field A are called external symmetries, while redundancies in the parametrisation of L are called internal symmetries. All internal symmetries are gauge symmetries, but one has to be more careful when dealing with the external symmetries. These external symmetries acting as $A \mapsto A^{\hat{h}}$ act on the new parametrisation by $L^{\hat{g}} \mapsto (L^{\hat{g}})^{\hat{h}} = L^{\hat{g}\hat{h}}$, and can be seen as right-acting symmetries on the edge mode $\hat{g} \mapsto \hat{g}\hat{h}$. Denoting the boundary conditions as $\hat{h}|_{\zeta=0} = h$ and $\hat{h}|_{\zeta=\infty} = \tilde{h}$, this acts on the edge mode as

$$g \mapsto gh, \quad \tilde{g} \mapsto \tilde{g}\tilde{h}, \quad \partial_+ h = 0, \quad \partial_- \tilde{h} = 0, \tag{2.138}$$

where the differential constraints ensure the boundary conditions are left invariant. There is also a bulk component of the external symmetries, but this is indistinguishable from the bulk internal symmetries since the edge mode plays no role in the bulk theory. Instead, we wish to look at the external symmetries at the poles of ω. Due to the differential constraints, these are semi-local physical symmetries with non-vanishing Noether charges. Unlike gauge symmetries, we should not attempt to fix these with gauge-fixing constraints, and instead they descend to physical symmetries of the boundary theory. We also have the internal symmetries that arise from the redundancy of the parametrisation $A = L^{\hat{g}}$. A is invariant under the transformation

$$L \mapsto L^{\check{h}}, \quad \hat{g} \mapsto \check{h}^{-1}\hat{g}. \tag{2.139}$$

These redundancies represent a gauge symmetry of the theory, which means we must fix the gauge by choosing a specific representative for the field content L and $\hat{g}$. We expect L to only have legs along Σ if it is to be a Lax for the boundary theory. Since ω already carries a $d\zeta$ leg, we also need to set

$$L_{\bar{\zeta}} = 0. \tag{2.140}$$

Now one must check if this fixing produces any residual symmetries. Since we can perform internal gauge transformations independent of the spectral parameter, we can fix

$$\hat{g}|_{\zeta=\infty} = \mathrm{id}. \tag{2.141}$$

The external symmetry parametrised by $\tilde{h}$ must now be combined with the internal gauge symmetry $\check{h} = \tilde{h}$ such that $\hat{g}|_{\zeta=\infty} = \mathrm{id}$ is unchanged. The surviving edge mode degree of freedom now transforms as

$$g \mapsto \tilde{h}^{-1}gh, \quad \partial_+ h = 0, \quad \partial_- \tilde{h} = 0. \tag{2.142}$$

To localize the 4dCS action to the boundary, we must solve some of the bulk equations of motion for L. Having fixed $L_{\bar{\zeta}} = 0$, we have two of the equations being the holomorphicity conditions

$$F_{\bar{\zeta}\pm} = \partial_{\bar{\zeta}} L_\pm = 0. \tag{2.143}$$

Since $L_\pm$ are holomorphic in ζ and $\mathbb{CP}^1$ is a compact manifold, Louiville's theorem states that they must be constant and hence independent of ζ. Using the gauge fixing conditions, we can solve the boundary conditions by

$$L_+ = -\partial_+ g g^{-1}, \quad L_- = 0. \tag{2.144}$$

This allows us to simplify the action, by writing it purely in terms of g and removing the first term, since nothing in L can contribute to the $d\zeta$ leg. Since we have fixed $\hat{g}|_{\zeta=\infty} = \mathrm{id}$, only the $\zeta = 0$ pole contributes to the boundary term. Showing this, we first write the action as

$$S_{\mathrm{4dCS}}[L, \hat{g}] = \int_\Sigma \left(\mathrm{tr}(L \wedge \mathrm{d}\hat{g}\hat{g}^{-1} - \int_{[0,1]} \mathrm{WZ}[\hat{g}] \right) \Bigg|_{\zeta=0}. \tag{2.145}$$

Now we write this in terms of g and evaluate it at $\zeta = 0$, giving us

$$S_{\text{WZW}}[g] = -\int_{\Sigma} \mathrm{d}^2x\, \mathrm{tr}(g^{-1}\partial_+ g\, g^{-1}\partial_- g) - \int_{\Sigma\times[0,1]} \mathrm{WZ}[g]. \tag{2.146}$$

This finishes the example of deriving a 2d field theory from 4dCS theory. The residual boundary symmetries of the edge mode correspond to semi-local symmetries of the WZW model. We also showed this model can be obtained from 3dCS theory. There is a connection here [13], which we can show by returning to the original form of the action and writing it as

$$S_{4\text{dCS}}[A] = \frac{1}{2\pi i}\int_{M_4} \left(\frac{\mathrm{d}\zeta}{\zeta}\right) \wedge \mathrm{CS}[A] = \frac{1}{2\pi i}\int_{M_4} \left(\frac{\mathrm{d}r}{r} + i\mathrm{d}\theta\right) \wedge \mathrm{CS}[A]. \tag{2.147}$$

Performing a dimensional reduction along the $U(1)$ direction by retaining only the θ-independent modes in the Fourier expansion, and imposing $A_\theta = 0$, the first term drops out and we are left with:

$$S_{4\text{dCS}}[A] = \frac{1}{2\pi}\int_{M_4} \mathrm{d}\theta \wedge \mathrm{CS}[A] = \underbrace{\int_{\Sigma\times[0,1]} \mathrm{CS}[A]}_{S_{3\text{dCS}}[A]}. \tag{2.148}$$

The endpoints here are identified with $\zeta = 0$ and $\zeta = \infty$, so A shares the boundary constraints of $A_+|_0 = 0$ and $A_-|_\infty = 0$. This is the setup to recover the WZW model from 3dCS theory [9].

2.4 Twistor theory

The material in this section closely follows Tim Adamo's lecture notes [14], which provide a comprehensive introduction to twistor theory and its applications. Twistor theory was introduced by Penrose [15] in the late 1960s as a novel geometric framework for addressing problems in space-time physics and, ultimately, quantum gravity. Although it led to a number of important developments, activity in the subject diminished by the late 1980s, and for many years twistors were used mainly in pure mathematics, particularly in the study of integrable systems and differential and algebraic geometry. Interest from the physics community was dramatically renewed in 2004, when Witten showed that twistor ideas could be combined with string-theoretic methods to compute tree-level scattering amplitudes in four-dimensional Yang–Mills theory [16], building on earlier insights of Nair [17]. Since then, twistor methods have become central to the study of novel structures in a range of perturbative quantum field theories.

2.4.1 Spinor and Twistor Basics

We begin in flat four-dimensional Minkowski space-time $\mathbb{M}$ with signature $(+,-,-,-)$. Before introducing twistor theory, we must go through complexification and spinor methods [5, 18]. The complexification $\mathbb{M}_{\mathbb{C}}$ of $(\mathbb{M}, \eta_{ab})$ allows coordinates $x^a \in \mathbb{C}$ while extending $\eta_{ab}(x)$ holomorphically, yielding $\mathbb{C}^4$ with line element

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2. \tag{2.149}$$

Different real slices recover Lorentzian $\mathbb{M}$, Euclidean $\mathbb{R}^4$, or split-signature $\mathbb{U}$ spaces, as described in Eq. (2.15).

2-spinors in Minkowski space

The spin group of complexified Minkowski space is $SO(4,\mathbb{C})$, which is locally isomorphic to $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, i.e. $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$. A vector in $\mathbb{M}_{\mathbb{C}}$ lives in the $(\frac{1}{2},\frac{1}{2})$ representation of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, so it can be represented by two $SL(2,\mathbb{C})$ indices. One can go between the vector and spinor indices using the Pauli matrices as follows:

$$v^{\alpha\dot{\alpha}} = \frac{\sigma_a^{\alpha\dot{\alpha}}}{\sqrt{2}} v^a = \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix}. \tag{2.150}$$

The un-dotted (negative chirality) spinor indices live in the $(\frac{1}{2}, 0)$ representation, while the dotted (positive chirality) ones live in the $(0, \frac{1}{2})$ representation. These indices can take values $\alpha = 0, 1$ and $\alpha = \dot{0}, \dot{1}$ respectively. This allows us to represent any number of vector indices with twice as many spinor indices. For example, a general contravariant tensor $T^{ab\ldots z}$ can be written as

$$T^{ab\ldots z} = T^{\alpha\dot{\alpha}\beta\dot{\beta}\ldots\zeta\dot{\zeta}}. \tag{2.151}$$

The spinor representation gives us the following equation for the norm of a given vector v^a:

$$\eta_{ab} v^a v^b = 2 \det(v^{\alpha\dot{\alpha}}). \tag{2.152}$$

This means that the spinor representation of any null vector has to have a rank less than two, since the determinant must be zero. This gives us

$$v^{\alpha\dot{\alpha}}_{null} = a^\alpha \tilde{a}^{\dot{\alpha}}. \tag{2.153}$$

The converse is also true, any $v^{\alpha\dot{\alpha}}$ that can be written as $a^\alpha \tilde{a}^{\dot{\alpha}}$ has a null vector representation. Hence, any pair of opposite chirality Weyl spinors corresponds to a null vector. For this to be useful we must define a way of raising and lowering indices in the spinor space. The object we use to do this is the $SL(2, \mathbb{C})$ invariant tensor, the 2d Levi-Civita tensor

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{\dot{\alpha}\dot{\beta}}. \tag{2.154}$$

The inverses for the un-dotted and dotted indices are defined by

$$\epsilon^{\alpha\beta}\epsilon_{\gamma\beta} = \delta^\alpha_\gamma, \quad \epsilon^{\alpha\beta}\epsilon_{\alpha\beta} = 2. \tag{2.155}$$

Since they are skew symmetric, we must fix a convention for how indices are raised and lowered to avoid sign errors. We choose "lower to the right, raise to the left" as our convention for both un-dotted and dotted indices:

$$a_\alpha \coloneqq a^\beta \epsilon_{\beta\alpha}, \quad b^\alpha \coloneqq \epsilon^{\alpha\beta} b_\beta. \tag{2.156}$$

The dual covector $v_{\alpha\dot{\alpha}}$ is then given by

$$v_{\alpha\dot{\alpha}} = v^{\beta\dot{\beta}}\epsilon_{\beta\alpha}\epsilon_{\dot{\beta}\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 - v^3 & -(v^1 + iv^2) \\ -(v^1 - iv^2) & v^0 + v^3 \end{pmatrix}. \tag{2.157}$$

It can then be seen that $v^{\alpha\dot{\alpha}} v_{\alpha\dot{\alpha}} = 2 \det(v^{\alpha\dot{\alpha}}) = \eta_{ab} v^a v^b$, and line element can be written as

$$\mathrm{d}s^2 = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\, \mathrm{d}x^{\alpha\dot{\alpha}} \mathrm{d}x^{\beta\dot{\beta}}. \tag{2.158}$$

The Levi-Civita symbol also gives us two inner products, one on the negative chirality space and one on the positive chirality one. These are given by

$$\langle \kappa\, \omega \rangle = \kappa^\alpha \omega_\alpha = \kappa^\alpha \omega^\beta \epsilon_{\beta\alpha}, \quad [\kappa\, \omega] = \kappa^{\dot{\alpha}} \omega_{\dot{\alpha}} = \kappa^{\dot{\alpha}} \omega^{\dot{\beta}} \epsilon_{\dot{\beta}\dot{\alpha}}. \tag{2.159}$$

This means we can write the inner product between two null vectors $v^a_{null} \equiv \kappa^\alpha \tilde{\kappa}^{\dot{\alpha}}$ and $w^a_{null} \equiv \omega^\alpha \tilde{\omega}^{\dot{\alpha}}$ as

$$v_{null} \cdot w_{null} = \langle \kappa\, \omega \rangle \, [\tilde{\kappa}\, \tilde{\omega}]. \tag{2.160}$$

Real slices and spinor conjugations

Having translated the metric geometry of $\mathbb{M}_\mathbb{C}$ into 2-spinors via

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \tag{2.161}$$

we identify reality conditions on $x^{\alpha\dot{\alpha}}$ compatible with each signature, each inducing a natural complex conjugation on spinor spaces [19]. Starting with Lorentzian Signature ($\mathbb{M}$), we require $x^{\alpha\dot{\alpha}} = (x^{\alpha\dot{\alpha}})^\dagger$ (Hermitian), where

$$(x^{\alpha\dot{\alpha}})^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{x}^0 + \bar{x}^3 & \bar{x}^1 - i\bar{x}^2 \\ \bar{x}^1 + i\bar{x}^2 & \bar{x}^0 - \bar{x}^3 \end{pmatrix}. \tag{2.162}$$

This exchanges un-dotted and dotted spinors:

$$\kappa^\alpha \mapsto \bar{\kappa}^{\dot{\alpha}} = (\bar{a}, \bar{b}), \quad \tilde{\omega}^{\dot{\alpha}} \mapsto \bar{\tilde{\omega}}^{\alpha} = (\bar{c}, \bar{d}), \tag{2.163}$$

for $\kappa^\alpha = (a, b)$, $\tilde{\omega}^{\dot{\alpha}} = (c, d)$. Real null vectors are $\kappa^\alpha \bar{\kappa}^{\dot{\alpha}}$. For Euclidean Signature ($\mathbb{R}^4$), we define

$$\hat{x}^{\alpha\dot{\alpha}} \coloneqq \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{x}^0 - \bar{x}^3 & -\bar{x}^1 + i\bar{x}^2 \\ -\bar{x}^1 - i\bar{x}^2 & \bar{x}^0 + \bar{x}^3 \end{pmatrix}. \tag{2.164}$$

Requiring $x^{\alpha\dot{\alpha}} = \hat{x}^{\alpha\dot{\alpha}}$ forces x^0 to be purely real and x^i, $i = 1, 2, 3$ to be purely imaginary, which in turn yields positive definite $x^2 = 2\det(x) = (x^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2$, $iy^i = x^i$. The induced quaternionic conjugation is

$$\kappa^\alpha \mapsto \hat{\kappa}^\alpha = (-\bar{b}, \bar{a}), \quad \tilde{\omega}^{\dot{\alpha}} \mapsto \hat{\tilde{\omega}}^{\dot{\alpha}} = (-\bar{d}, \bar{c}), \tag{2.165}$$

satisfying $\hat{\hat{\kappa}}^\alpha = -\kappa^\alpha$. The nature of this conjugation means that there are no real null vectors in Euclidean space. For ultrahyperbolic signature ($\mathbb{U}$), we require $x^{\alpha\dot{\alpha}} = \overline{x^{\alpha\dot{\alpha}}}$ (ordinary complex conjugate), yielding $x^2 = (x^0)^2 + (y^2)^2 - (x^1)^2 - (x^3)^2$. Conjugation preserves chirality:

$$\kappa^\alpha \mapsto \bar{\kappa}^\alpha = (\bar{a}, \bar{b}), \quad \tilde{\omega}^{\dot{\alpha}} \mapsto \bar{\tilde{\omega}}^{\dot{\alpha}} = (\bar{c}, \bar{d}), \tag{2.166}$$

so 2-spinors have real components ($\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$). Null vectors are $\kappa^\alpha \tilde{\kappa}^{\dot{\alpha}}$ with real spinors.

Twistor space

Now that we have defined spinors, we are ready to introduce the twistor correspondence. First we define $\mathbb{CP}^3$ to be the 3d complex projective space, which is the space of all straight lines through the origin in $\mathbb{C}^4$. $\mathbb{CP}^3$ can be described by homogeneous coordinates $Z^A = (Z^0, Z^1, Z^2, Z^3)$, which are never all vanishing, and are equivalent up to rescaling

$$(Z^0, Z^1, Z^2, Z^3) \neq (0, 0, 0, 0), \quad rZ^A \sim Z^A, \quad \forall r \in \mathbb{C}^*, \tag{2.167}$$

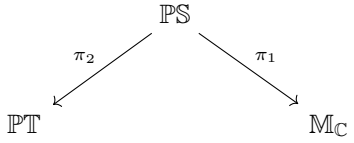
where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. The equivalence relation of rescaling means this space only has 3 degrees of freedom instead of 4. This can be seen by the fact that one can always set one of the coordinates to 1. For example, if we cover the manifold with patches $U_i = \{Z^A \in \mathbb{C}^4 | Z^i \neq 0\}$, then we can define local coordinates Z^j/Z^i, where $j \neq i$. The twistor space $\mathbb{PT}$ of $\mathbb{M}_\mathbb{C}$ is given by an open subset of $\mathbb{CP}^3$. On $\mathbb{PT}$, we can write the 4 complex coordinates as two Weyl spinors of opposite chirality:

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha), \tag{2.168}$$

where each spinor carries the same weight with respect to rescaling, i.e. this can just be thought of as new notation for the same object at this point. The tricky part is defining a relation between $\mathbb{PT}$ and space-time. This is a non-local relationship and is referred to as the twistor correspondence. For $\mathbb{M}_\mathbb{C}$, the twistor correspondence is given by the algebraic relation:

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \lambda_\alpha. \tag{2.169}$$

These equations are known as the incidence relations, and are the root of why twistor theory is so interesting. More formally, this is presented as a double fibration of the projective spinor bundle over $\mathbb{M}_\mathbb{C}$ and $\mathbb{PT}$, where $\mathbb{PS}$ has coordinates $(x^{\alpha\dot{\alpha}}, \lambda_\beta)$ with $\lambda \sim r\lambda$ for $r \in \mathbb{C}^*$. This means that



on $\mathbb{PS}$, λ_β acts as a homogeneous coordinate on $\mathbb{CP}^1 \cong S^1$. So $\mathbb{PS} \cong \mathbb{M}_\mathbb{C} \times \mathbb{CP}^1$, and we have the maps

$$\begin{aligned} &\pi_1 : \mathbb{PS} \to \mathbb{M}_\mathbb{C}, && \pi_2 : \mathbb{PS} \to \mathbb{PT}, \\ &(x^{\alpha\dot{\alpha}}, \lambda_\beta) \mapsto x^{\alpha\dot{\alpha}}, && (x^{\alpha\dot{\alpha}}, \lambda_\beta) \mapsto (x^{\beta\dot{\alpha}}\lambda_\beta, \lambda_\alpha). \end{aligned} \tag{2.170}$$

For us, it is sufficient to think about twistor space just in terms of the incidence relations. Say now if we fix a point $x \in \mathbb{M}_{\mathbb{C}}$, what does this correspond to in twistor space? Neglecting for a moment the projective nature of twistor space, $\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha$ defines a complex plane $\mathbb{C}^2 \subset \mathbb{C}^4$. Now applying the projective nature to this, the incidence relations define a $\mathbb{CP}^1 \subset \mathbb{PT}$, so each point $x \in \mathbb{M}_{\mathbb{C}}$ corresponds to a linearly and holomorphically embedded Riemann sphere in $\mathbb{PT}$. A Riemann sphere corresponding to a point x is denoted as X, and we often refer to these spheres as "lines". This is our first exposure to non-locality in the twistor correspondence, a point in Minkowski space corresponds to an extended object in twistor space. Looking at the reverse, what does a point in twistor space correspond to? Viewing a point as the intersection of two lines X and Y, we have

$$\begin{aligned} X \cap Y = \{Z \in \mathbb{PT}\} &\implies \mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha \quad \text{and} \quad \mu^{\dot{\alpha}} = y^{\alpha\dot{\alpha}}\lambda_\alpha \\ &\implies (x-y)^{\alpha\dot{\alpha}}\lambda_\alpha = 0 \\ &\implies (x-y)^{\alpha\dot{\alpha}} \propto \lambda^\alpha \\ &\implies (x-y)^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}. \end{aligned} \tag{2.171}$$

This means that lines in twistor space only intersect if their corresponding points in Minkowski space are null separated. By finding all of the lines that intersect at a point Z in twistor space, one can associate a point in twistor space with an α-plane in complex Minkowski space. An α-plane is a totally null complex 2-plane in $\mathbb{M}_{\mathbb{C}}$: every tangent vector to the plane is of the form $\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$, where λ^α is fixed and $\tilde{\lambda}^{\dot{\alpha}}$ varies freely, giving two degrees of freedom. Since all tangent vectors are null and mutually orthogonal, the plane is totally null, and its tangent bivectors are self-dual.

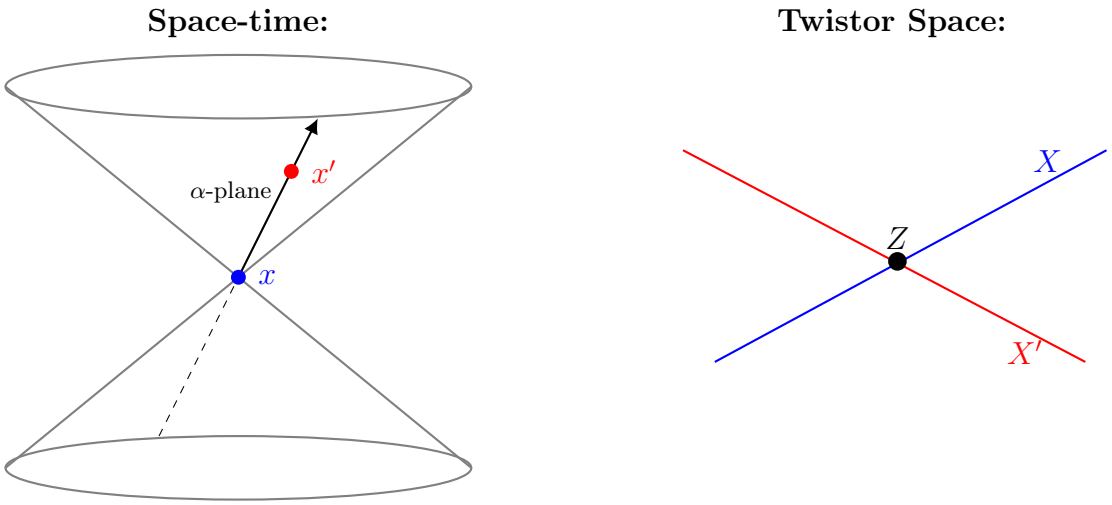


Figure 2.2: Null-separated points x, x' on light cone correspond to lines X, X' intersecting at $Z \in \mathbb{PT}$.

Points in $\mathbb{M}_{\mathbb{C}}$ can be represented by two points Z_1 and Z_2 in $\mathbb{PT}$ which lie on the line X. Let's compute the bi-twistor $X^{AB} = Z_1^{[A}Z_2^{B]}$.

$$\begin{aligned} X^{AB} = Z_1^{[A}Z_2^{B]} &= Z_1^A Z_2^B - Z_2^A Z_1^B \\ &= \begin{pmatrix} \mu_1^{\dot{\alpha}}\mu_2^{\dot{\beta}} - \mu_2^{\dot{\alpha}}\mu_1^{\dot{\beta}} & \mu_1^{\dot{\alpha}}\lambda_{2,\beta} - \mu_2^{\dot{\alpha}}\lambda_{1,\beta} \\ \lambda_{1,\alpha}\mu_2^{\dot{\beta}} - \lambda_{2,\alpha}\mu_1^{\dot{\beta}} & \lambda_{1,\alpha}\lambda_{2,\beta} - \lambda_{2,\alpha}\lambda_{1,\beta} \end{pmatrix}. \end{aligned} \tag{2.172}$$

We now wish to simplify these terms. To start off, we use antisymmetry to compute

$$\lambda_{1,\alpha}\lambda_{2,\beta} - \lambda_{2,\alpha}\lambda_{1,\beta} = \langle\lambda_1\,\lambda_2\rangle\epsilon_{\alpha\beta}. \tag{2.173}$$

Now we use the incidence relations to compute the top left term

$$\begin{aligned} \mu_1^{\dot{\alpha}}\mu_2^{\dot{\beta}} - \mu_2^{\dot{\alpha}}\mu_1^{\dot{\beta}} &= x^{\alpha\dot{\alpha}}x^{\beta\dot{\beta}}(\lambda_{1,\alpha}\lambda_{2,\beta} - \lambda_{2,\alpha}\lambda_{1,\beta}) \\ &= x^{\alpha\dot{\alpha}}x^{\beta\dot{\beta}}\langle\lambda_1\,\lambda_2\rangle\epsilon_{\alpha\beta} \\ &= x^{\alpha\dot{\alpha}}x_\alpha^{\dot{\beta}}\langle\lambda_1\,\lambda_2\rangle. \end{aligned} \tag{2.174}$$

Since this is antisymmetric in $\dot{\alpha}$ and $\dot{\beta}$, it must be proportional to a Levi-Civita tensor, so we contract both sides of the following equation:

$$\begin{aligned} x^{\alpha\dot{\alpha}} x_{\alpha}^{\dot{\beta}} \langle \lambda_1 \, \lambda_2 \rangle &= C \epsilon^{\dot{\alpha}\dot{\beta}} \\ \implies x^2 \langle \lambda_1 \, \lambda_2 \rangle &= 2C \\ \implies \mu_1^{\dot{\alpha}} \mu_2^{\dot{\beta}} - \mu_2^{\dot{\alpha}} \mu_1^{\dot{\beta}} &= \frac{1}{2} x^2 \epsilon^{\dot{\alpha}\dot{\beta}} \langle \lambda_1 \, \lambda_2 \rangle. \end{aligned} \tag{2.175}$$

There are then two terms left to compute:

$$\begin{aligned} \mu_1^{\dot{\alpha}} \lambda_{2,\beta} - \mu_2^{\dot{\alpha}} \lambda_{1,\beta} &= x^{\alpha\dot{\alpha}} (\lambda_{1,\alpha} \lambda_{2,\beta} - \lambda_{2,\alpha} \lambda_{1,\beta}) = x^{\alpha\dot{\alpha}} \langle \lambda_1 \, \lambda_2 \rangle \epsilon_{\alpha\beta} = x_{\beta}^{\dot{\alpha}} \langle \lambda_1 \, \lambda_2 \rangle, \\ \lambda_{1,\alpha} \mu_2^{\dot{\beta}} - \lambda_{2,\alpha} \mu_1^{\dot{\beta}} &= x^{\beta\dot{\beta}} (\lambda_{1,\alpha} \lambda_{2,\beta} - \lambda_{2,\alpha} \lambda_{1,\beta}) = x^{\beta\dot{\beta}} \langle \lambda_1 \, \lambda_2 \rangle \epsilon_{\alpha\beta} = -x_{\alpha}^{\dot{\beta}} \langle \lambda_1 \, \lambda_2 \rangle. \end{aligned} \tag{2.176}$$

Substituting these expressions, we get the following form for the bi-twistor:

$$X^{AB} = \langle \lambda_1 \, \lambda_2 \rangle \begin{pmatrix} \frac{1}{2} x^2 \epsilon^{\dot{\alpha}\dot{\beta}} & x_{\beta}^{\dot{\alpha}} \\ -x_{\alpha}^{\dot{\beta}} & \epsilon_{\alpha\beta} \end{pmatrix}. \tag{2.177}$$

2.4.2 Twistor Geometry

Now we know that twistor space relates non-locally to complexified Minkowski space. Points in $\mathbb{M}_{\mathbb{C}}$ correspond to lines in $\mathbb{PT}$, and points in $\mathbb{PT}$ correspond to α-planes in $\mathbb{M}_{\mathbb{C}}$. The conformal structure of space-time manifests through the holomorphic geometry of lines in twistor space. Twistor space of complexified Minkowski space $\mathbb{M}_{\mathbb{C}}$ is the total space of the holomorphic vector bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1, \tag{2.178}$$

where $\mathcal{O}(1)$ is the standard holomorphic line bundle of degree $+1$ over the Riemann sphere $\mathbb{CP}^1$. Taking two copies and forming $\mathcal{O}(1) \oplus \mathcal{O}(1)$ gives a rank-2 holomorphic vector bundle whose fibre over each point of $\mathbb{CP}^1$ is $\mathbb{C}^2$, but twisted according to the $\mathcal{O}(1)$ bundle structure. This bundle structure is compatible with the incidence relations (2.169) established in Section 2.4.1, where each point $x^{\alpha\dot{\alpha}} \in \mathbb{M}_{\mathbb{C}}$ corresponds to a holomorphic line in $\mathbb{PT}$. We wish to examine how space-time structures translate into geometric structures on twistor space.

Reality structures

We previously looked at reality conditions and subsequent notions of complex conjugation for different real slices of $\mathbb{M}_{\mathbb{C}}$. Now, we wish to see what conditions we need to impose on $\mathbb{PT}$ for it to correspond to different sections of $\mathbb{M}_{\mathbb{C}}$. We start with Lorentzian signature as before, where complex conjugation of a given twistor $Z^A = (\mu^{\dot{\alpha}}, \lambda_{\alpha})$ is given by

$$(\mu^{\dot{\alpha}}, \lambda_{\alpha}) \mapsto (\bar{\lambda}_{\dot{\alpha}}, \bar{\mu}^{\alpha}). \tag{2.179}$$

This can be interpreted as a kind of "duality" on twistor space. To make thing easier for us, we shift the incidence relations by a factor of i:

$$\mu^{\dot{\alpha}} = i \, x^{\alpha\dot{\alpha}} \lambda_{\alpha}. \tag{2.180}$$

Now let $\mathbb{PT}^{\vee}$ be the same open subset as $\mathbb{PT} \subset \mathbb{CP}^3$ but now with homogeneous coordinates $W_A = (\tilde{\lambda}_{\dot{\alpha}}, \tilde{\mu}^{\alpha})$. Points in this dual twistor space obey the following incidence relations:

$$\tilde{\mu}^{\dot{\alpha}} = -i \, x^{\alpha\dot{\alpha}} \tilde{\lambda}_{\alpha}. \tag{2.181}$$

There is a natural inner product between twistor space and its dual given by

$$Z \cdot W = Z^A W_A = [\mu \, \tilde{\lambda}] + \langle \tilde{\mu} \, \lambda \rangle. \tag{2.182}$$

So, the Lorentzian reality structure induces a complex conjugation that sends a twistor to a dual twistor whose components are complex conjugates of the original twistor. This induces an inner product of the form

$$Z \cdot \bar{Z} = [\mu \, \bar{\lambda}] + \langle \bar{\mu} \, \lambda \rangle. \tag{2.183}$$

Given a line $X \in \mathbb{PT}$, how can we check that the corresponding $x^{\alpha\dot{\alpha}}$ is valued in real Minkowski space? Let Z be a point in twistor space on the line X. It follows that

$$Z \cdot \bar{Z} = i(x - x^\dagger)^{\alpha\dot{\alpha}} \lambda_\alpha \lambda_{\dot{\alpha}}. \tag{2.184}$$

But the reality condition for real Minkowski space is that $x^{\alpha\dot{\alpha}} = (x^{\alpha\dot{\alpha}})^\dagger$. Hence any line X associated with a point of real Minkowski space must be contained in the space of null twistors, given by

$$\mathbb{PN} = \{Z \in \mathbb{PT} \,|Z \cdot \bar{Z} = 0\}. \tag{2.185}$$

A point in $\mathbb{PN}$ corresponds to a real null geodesic, $\lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$ in $\mathbb{M}$. This is the line where the complex α-plane intersects the real slice $\mathbb{M} \subset \mathbb{M}_{\mathbb{C}}$. A visualization of this can be seen in Fig. 2.3.

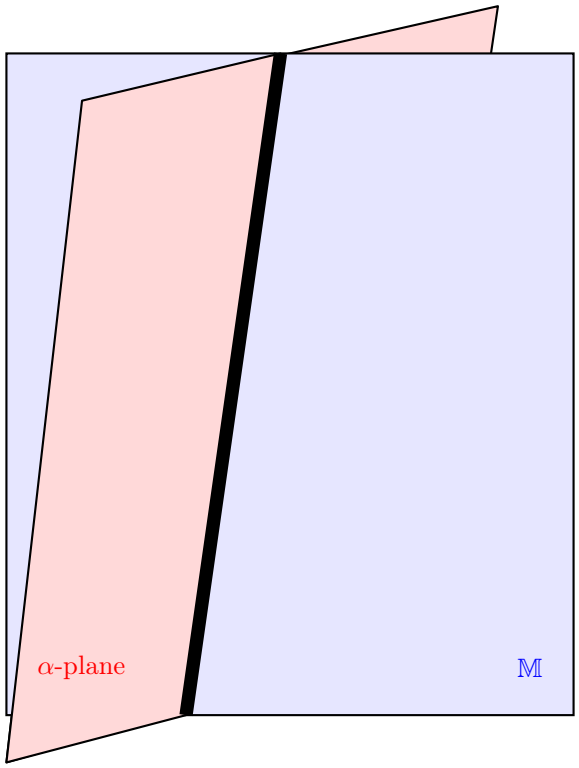


Figure 2.3: Intersection between α-plane and real Minkowski space $\mathbb{M} \subset \mathbb{M}_{\mathbb{C}}$.

Turning now to Euclidean signature, the real slice $\mathbb{R}^4$ inside $\mathbb{M}_{\mathbb{C}}$ induces quaternionic conjugation on spinors:

$$\kappa^\alpha = (a, b) \mapsto \hat{\kappa}^\alpha = (-\bar{b}, \bar{a}), \quad \tilde{\omega}^{\dot{\alpha}} = (c, d) \mapsto \hat{\tilde{\omega}}^{\dot{\alpha}} = (-\bar{d}, \bar{c}). \tag{2.186}$$

On twistor space, this yields an involution $\sigma : \mathbb{PT} \to \mathbb{PT}$ via

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha) \mapsto \hat{Z}^A = (\hat{\mu}^{\dot{\alpha}}, \hat{\lambda}_\alpha). \tag{2.187}$$

Since $\sigma^2 = -\mathrm{id}$, no fixed points exist in $\mathbb{PT}$, corresponding to α-planes missing the real slice $\mathbb{R}^4$ (no real null geodesics). Lines preserved by σ take the form $X^{AB} = Z^{[A}\hat{Z}^{B]}$, mapping each $Z \in \mathbb{PT}$ to a unique $x \in \mathbb{R}^4$ via the line through Z and $\hat{Z}$. This defines a $\mathbb{CP}^1$-fibration $\mathbb{PT} \to \mathbb{R}^4$, explicitly

$$x^{\alpha\dot{\alpha}} = \frac{\hat{\mu}^{\dot{\alpha}} \lambda^\alpha - \mu^{\dot{\alpha}} \hat{\lambda}^\alpha}{\langle \lambda \hat{\lambda} \rangle}, \tag{2.188}$$

compatible with incidence $x^{\alpha\dot{\alpha}} \lambda_\alpha = \mu^{\dot{\alpha}}$. This fibration is encoded by the inclusion map

$$\iota_x : \mathbb{CP}^1_x \hookrightarrow \mathbb{PT}, \quad \lambda_\alpha \mapsto (\mu^{\dot{\alpha}}, \lambda_\alpha) = (x^{\beta\dot{\alpha}} \lambda_\beta, \lambda_\alpha), \tag{2.189}$$

which sends each fibre coordinate λ_α to the twistor $(\mu^{\dot\alpha}, \lambda_\alpha)$ with ω^A determined by the incidence relation. Hence, each space-time point $x^{\alpha\dot\alpha} \in \mathbb{M}_{\mathbb{C}}$ corresponds to a unique holomorphic line in $\mathbb{PT}$. We can think of this fibration as a smooth identification between $\mathbb{PT}$ and the left-handed projective spinor bundle $\mathbb{PS}^+$, so that Euclidean twistor space is $\mathbb{R}^4 \times \mathbb{CP}^1 \cong \mathbb{PS}^+$. For ultrahyperbolic space, ordinary complex conjugation $\overline{Z^A} = (\bar\mu^{\dot\alpha}, \bar\lambda_\alpha)$ preserves a real locus $\mathbb{PT}_{\mathbb{R}} \subset \mathbb{RP}^3$, where lines $X \subset \mathbb{PT}_{\mathbb{R}}$ satisfy incidence trivially since $x = \bar x$. Split signature uses manifestly real $\mathrm{SL}(2,\mathbb{R})$ spinors.

Complex Structures

A complex structure on $\mathbb{PT}$ determines a conformal structure on spacetime and vice versa, manifest in the geometry where the conformal structure of $\mathbb{M}_{\mathbb{C}}$ arises from intersections of holomorphic lines in twistor space. An almost complex structure on a manifold M is a linear map $J : TM \to TM$ that satisfies $J^2 = -\mathrm{id}$. It is integrable (a true complex structure) if the Nijenhuis tensor vanishes:

$$(N_J)^k_{ij} = J^l_j \partial_l J^k_i - J^l_i \partial_l J^k_j + J^k_l (\partial_i J^l_j - \partial_j J^l_i) = 0. \tag{2.190}$$

Equivalently, the complexified tangent space can be decomposed into $\pm i$ eigenspaces of J,

$$TM_{\mathbb{C}} = T^{(1,0)}_M \oplus T^{(0,1)}_M, \tag{2.191}$$

inducing a decomposition of k-forms:

$$\Omega^k(M)_{\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}(M). \tag{2.192}$$

The Dolbeault operator $\bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$, defined as $\bar\partial|_{\Omega^{p,q}(M)} = \rho^{p,q+1} \circ d$, characterizes holomorphy ($\bar\partial f = 0$). Integrability requires $\bar\partial^2 = 0$. Given a notion of complex conjugation, the complex structure on $\mathbb{PT} \subset \mathbb{CP}^3$ is given by

$$\bar\partial = \mathrm{d}\bar Z^{\bar A} \frac{\partial}{\partial \bar Z^{\bar A}}. \tag{2.193}$$

Under Euclidean reality conditions, explicit bases are:

$$\begin{aligned} T^{0,1}_{\mathbb{PT}} &= \mathrm{span}\left\{ \bar\partial_0 = \langle \lambda\, \hat\lambda\rangle \lambda^\alpha \frac{\partial}{\partial \hat\lambda^\alpha},\ \hat\partial_{\dot\alpha} = \lambda^\alpha \frac{\partial}{\partial x^{\alpha\dot\alpha}} \right\}, \\ \Omega^{0,1}(\mathbb{PT}) &= \mathrm{span}\left\{ \bar e^0 = \frac{\langle \hat\lambda\, \mathrm{d}\hat\lambda\rangle}{\langle \lambda\, \hat\lambda\rangle^2},\ \hat e^{\dot\alpha} = \frac{\hat\lambda_\alpha \mathrm{d}x^{\alpha\dot\alpha}}{\langle \lambda\, \hat\lambda\rangle} \right\}. \end{aligned} \tag{2.194}$$

The complex structure is $\bar\partial = \bar e^0 \bar\partial_0 + \bar e^{\dot\alpha} \bar\partial_{\dot\alpha}$, compatible with incidence relations and integrable: $\bar\partial^2 = 0$. There is a canonical holomorphic 3-form on $\mathbb{PT}$ with holomorphic weight 4 inherited from its natural inclusion into $\mathbb{CP}^3$:

$$D^3 Z = \frac{\epsilon_{ABCD} Z^A \mathrm{d}Z^B \wedge \mathrm{d}Z^C \wedge \mathrm{d}Z^D}{4!} = \frac{\langle \lambda\, \mathrm{d}\lambda\rangle \wedge \lambda_\alpha \lambda_\beta\, \mathrm{d}^2 x^{\alpha\beta}}{2} \in \Omega^{3,0}(\mathbb{PT}, \mathcal{O}(4)). \tag{2.195}$$

The convention we use throughout this dissertation is

$$\mathrm{d}^2 x^{\alpha\beta} = \frac{1}{2} \epsilon_{\dot\beta\dot\alpha} \mathrm{d}x^{\alpha\dot\alpha} \wedge \mathrm{d}x^{\beta\dot\beta}. \tag{2.196}$$

This 3-form will serve as the starting point for defining the holomorphic Chern–Simons action on twistor space in Section 3.1.

Conformal Structures

Null cones at points in $\mathbb{M}_{\mathbb{C}}$ (or real slices) are encoded by twistor line intersections in $\mathbb{PT}$, capturing the conformal class of conformally flat metrics but insensitive to boundary conditions at infinity (e.g., Minkowski vs. dS_4). The complexified conformal group $\mathrm{SL}(4,\mathbb{C})$ acts linearly on $\mathbb{PT}$ via holomorphic generators

$$T^A_B = Z^A \frac{\partial}{\partial Z^B}. \tag{2.197}$$

In standard notation, the generators of the conformal group are written in twistor space as:

$$\begin{aligned} P_{\alpha\dot{\alpha}} &= \lambda^{\alpha}\frac{\partial}{\partial\mu^{\dot{\alpha}}}, & J_{\alpha\beta} &= \lambda_{(\alpha}\frac{\partial}{\partial\lambda^{\beta)}}, \\ \tilde{J}_{\dot{\alpha}\dot{\beta}} &= \mu_{(\dot{\alpha}}\frac{\partial}{\partial\mu^{\dot{\beta})}}, & K^{\alpha\dot{\alpha}} &= \mu^{\dot{\alpha}}\frac{\partial}{\partial\lambda_{\alpha}}, \\ D &= \frac{1}{2}\left(\lambda_{\alpha}\frac{\partial}{\partial\lambda^{\alpha}} - \mu^{\dot{\alpha}}\frac{\partial}{\partial\mu^{\dot{\alpha}}}\right). \end{aligned} \tag{2.198}$$

These realize the conformal algebra, but $\mathbb{PT}$ cannot distinguish conformally flat spacetimes without additional structure breaking conformal invariance: the structure of conformal infinity. A conformally flat metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^{\alpha\dot{\alpha}}\mathrm{d}x_{\alpha\dot{\alpha}}}{f(x)^2} \tag{2.199}$$

requires a projectively invariant line element on the Klein quadric $Q = \{X \in \mathbb{CP}^5 \mid X^2 = 0\}$ (parametrizing lines in $\mathbb{PT}$ via $X^{AB} = Z_1^{[A}Z_2^{B]}$):

$$\mathrm{d}s^2 = \epsilon_{ABCD}\frac{\mathrm{d}X^{AB}\mathrm{d}X^{CD}}{(I_{EF}X^{EF})^2}, \tag{2.200}$$

singular on the hypersurface $I_{AB}X^{AB} = 0$, points at infinity. Thus, the infinity twistor I_{AB} encodes the conformal structure; $\mathbb{PT}$ excludes lines with $I_{AB}X^{AB} = 0$. For flat $\mathbb{M}_{\mathbb{C}}$, take

$$I_{AB} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{\alpha\beta} \end{pmatrix}, \tag{2.201}$$

yielding $I_{AB}X^{AB} = \langle\lambda_1\lambda_2\rangle$ and the Minkowski metric. Lines with $I_{AB}X^{AB} = 0$ have $\lambda^{\alpha} = 0$, forcing infinite $x^{\alpha\dot{\alpha}}$ via incidence—conformal infinity. Moreover, $I^2 = 0$ implies I is a line in $\mathbb{PT}$ (spacelike infinity i^0); lines meeting I are null infinity $\mathcal{I}^{\pm}$. The infinity twistor will reappear in Section 3.2, where it breaks conformal invariance of the WZW_4 model down to the Poincaré group.

2.4.3 The Penrose Transform

Twistor theory seems interesting as a mathematical excursion, but it is useful from a physics perspective also. One of the oldest use cases for twistor theory is for constructing solutions to free field equations. Every massless free field of integer or half-integer spin in 4d flat space-time can be represented on twistor space by something known as a cohomology class, which is a differential form which obeys some simple differential equations.

Zero-rest-mass fields

In physics one often deals with free fields. We often think of these fields in terms of gage potentials such as a massless scalar Φ for the Klein-Gordon field (spin 0), A_a for the Maxwell field (spin 1), and the linearized metric h_{ab} for spin 2. However, different gauge potentials can describe the same physical field. For instance in Maxwell theory the gauge potential is unique only up to a $U(1)$ gauge transformation, and similar transformations exist for different fields. The objects which are invariant under these transformations are the linearized curvature tensors associated with each field. In 4d, certain structures underlying these curvature tensors become manifest when working in the 2-spinor formalism. This allows us to write the free field equations for massless fields in terms of these underlying structures. To see this let's first write everything in terms of spinor indices:

$$A_a \leftrightarrow A_{\alpha\dot{\alpha}}, \quad F_{ab} \leftrightarrow F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \partial_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}}. \tag{2.202}$$

The field strength here is clearly antisymmetric under the exchange of $(\alpha\dot{\alpha}) \leftrightarrow (\beta\dot{\beta})$ the same way it is antisymmetric under the exchange of $a \leftrightarrow b$. This means that any contribution to F is either symmetric under the exchange of $\alpha \leftrightarrow \beta$ and antisymmetric under the exchange of $\dot{\alpha} \leftrightarrow \dot{\beta}$ or the reverse. Anything that is antisymmetric under the exchange of $\alpha \leftrightarrow \beta$ must proportional to $\epsilon_{\alpha\beta}$,

and the equivalent holds for antisymmetry in dotted indices. Hence, we can decompose the field strength as

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \frac{1}{2}\epsilon_{\alpha\beta}F^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}} + \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha}{}^{\dot{\gamma}}{}_{\beta\dot{\gamma}}. \tag{2.203}$$

Seeing that these contracted parts of F are symmetric in their remaining free spinor indices, we define

$$\tilde{F}_{\dot{\alpha}\dot{\beta}} = \tilde{F}_{(\dot{\alpha}\dot{\beta})} \coloneqq \frac{1}{2}F^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}}\,, \quad F_{\alpha\beta} = F_{(\alpha\beta)} \coloneqq \frac{1}{2}F_{\alpha}{}^{\dot{\gamma}}{}_{\beta\dot{\gamma}}\,, \tag{2.204}$$

which will be referred to as the self-dual and anti-self-dual parts of the field strength, respectively. This gives us the expression

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta}. \tag{2.205}$$

To see why these are called the self-dual (SD) and anti-self-dual (ASD) parts of the field strength one can simply contract the field strength with the epsilon tensor as follows

$$\begin{aligned} &\epsilon^{abcd} \leftrightarrow \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\epsilon^{\dot{\alpha}\dot{\delta}}\epsilon^{\dot{\beta}\dot{\gamma}} - \epsilon^{\alpha\delta}\epsilon^{\beta\gamma}\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\delta}}, \\ &\frac{1}{2}\epsilon^{abcd}F_{ab} \leftrightarrow \epsilon^{\gamma\delta}\tilde{F}^{\dot{\gamma}\dot{\delta}} - \epsilon^{\dot{\gamma}\dot{\delta}}F^{\gamma\delta}. \end{aligned} \tag{2.206}$$

So $\tilde{F}_{\dot{\alpha}\dot{\beta}}$ and $F_{\alpha\beta}$ have eigenvalues $+1$ and -1 under the duality operation, respectively. Written in terms of this decomposition, the Maxwell and equations and Bianchi identities for the field strength are given by

$$\begin{aligned} \partial^{\dot{\alpha}}_{\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} + \partial^{\alpha}_{\dot{\beta}}\tilde{F}_{\alpha\beta} = 0, \\ \partial^{\dot{\alpha}}_{\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} - \partial^{\alpha}_{\dot{\beta}}\tilde{F}_{\alpha\beta} = 0, \end{aligned} \tag{2.207}$$

respectively. As can be seen by these equations either a purely SD or ASD field strength automatically obeys the Maxwell equations, since any field strength F must obey the Bianchi identity by definition. These SD and ASD sectors are precisely the two on-shell photon polarizations in 4d, and are referred to as positive or negative helicity. A summary of these sectors are given below by what are known as zero-rest-mass (z.r.m.) equations.

$$\text{SD Sector (postive helicity):} \quad \partial^{\dot{\alpha}}_{\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} = 0, \tag{2.208}$$

$$\text{ASD Sector (negative helicity):} \quad \partial^{\alpha}_{\dot{\beta}}F_{\alpha\beta} = 0. \tag{2.209}$$

A similar decomposition can be done for any integer of half-integer spin field. For instance, for a spin-2 field, the Riemann curvature tensor can be split up as follows [5]:

$$\begin{aligned} R_{abcd} \longleftrightarrow\ &\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} + \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}}\Psi_{\alpha\beta\gamma\delta} + \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\gamma\delta}\Phi_{\alpha\beta\dot{\gamma}\dot{\delta}} + \epsilon_{\alpha\beta}\epsilon_{\dot{\gamma}\dot{\delta}}\Phi_{\dot{\alpha}\dot{\beta}\gamma\delta} \\ &+ \frac{R}{12}\left(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}} - \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}\epsilon_{\dot{\alpha}\dot{\delta}}\epsilon_{\dot{\beta}\dot{\gamma}}\right). \end{aligned} \tag{2.210}$$

Here, both $\tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$ and $\Psi_{\alpha\beta\gamma\delta}$ are totally symmetric and represent the SD and ASD parts of the Weyl curvature, $\Phi_{\alpha\beta\dot{\gamma}\dot{\delta}}$ represents the trace-free Ricci curvature, and R is the Ricci scalar. The vacuum Einstein equations enforce $R = 0$, and the Bianchi identity is then given by

$$\nabla_{[a}R_{bc]de} = 0 \leftrightarrow \epsilon_{\gamma\delta}\nabla^{\dot{\alpha}}_{\beta}\tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} - \epsilon_{\dot{\gamma}\dot{\delta}}\nabla^{\alpha}_{\dot{\beta}}\Psi_{\alpha\beta\gamma\delta} = 0. \tag{2.211}$$

Linearizing these by replacing the covariant derivatives with partial ones gives the two z.r.m. equations for helicity ± 2.

$$\text{SD Sector } (+2 \text{ helicity}): \quad \partial^{\dot{\alpha}}_{\beta}\tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = 0, \tag{2.212}$$

$$\text{ASD Sector } (-2 \text{ helicity}): \quad \partial^{\alpha}_{\dot{\beta}}\Psi_{\alpha\beta\gamma\delta} = 0. \tag{2.213}$$

For a general z.r.m. field of helicity h, the z.r.m. equations are as follows:

$$h > 0 \quad \tilde{\phi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2|h|}}, \quad \partial^{\beta\dot{\alpha}_1}\tilde{\phi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2|h|}} = 0, \tag{2.214}$$

$$h = 0 \quad \Phi, \quad \Box\Phi = \partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\Phi = 0, \tag{2.215}$$

$$h < 0 \quad \phi_{\alpha_1\cdots\alpha_{2|h|}}, \quad \partial^{\alpha_1\dot{\beta}}\phi_{\alpha_1\cdots\alpha_{2|h|}} = 0. \tag{2.216}$$

These z.r.m. equations are also conformally invariant. To see this, consider a conformal rescaling of $\mathbb{M}_{\mathbb{C}}$,

$$\eta_{ab} \to \Omega^2(x)\eta_{ab}. \tag{2.217}$$

In spinor notation, where $\eta_{ab} \leftrightarrow \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}$, this is equivalent to

$$\epsilon_{\alpha\beta} \to \Omega(x)\epsilon_{\alpha\beta}, \quad \epsilon_{\dot{\alpha}\dot{\beta}} \to \Omega(x)\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{2.218}$$

Looking at Eq. (2.205), it is easy to see that the z.r.m. fields must then transform as Ω^{-1}, since F_{ab} is conformally invariant. Similar arguments can be used to show this is actually true for all z.r.m. fields. For concreteness, consider

$$\partial^{\alpha\dot{\alpha}}\phi_{\alpha\beta\dots\gamma} = 0. \tag{2.219}$$

Using the fact that the epsilon tensors have conformal weight 1, $\phi_{\alpha\beta\dots\gamma}$ has conformal weight -1, and $\partial_{\alpha\dot{\alpha}}\epsilon_{\beta\gamma} = 0$, one obtains that

$$\Omega\hat{\nabla}_{\alpha\dot{\alpha}}\hat{\phi}_{\beta\dots\gamma} = \Omega\hat{\nabla}_{\alpha\dot{\alpha}}\left(\Omega^{-1}\phi_{\beta\dots\gamma}\right) = \partial_{\alpha\dot{\alpha}}\phi_{\beta\dots\gamma} - \Upsilon_{\dot{\alpha}\alpha}\phi_{\beta\dots\gamma} - \Upsilon_{\dot{\alpha}\beta}\phi_{\alpha\dots\gamma} - \dots - \Upsilon_{\dot{\alpha}\gamma}\phi_{\beta\dots\alpha}, \tag{2.220}$$

where hatted indices indicate objects in the conformally rescaled metric, and

$$\Upsilon_{\dot{\alpha}\alpha} \coloneqq \frac{1}{k}\Omega^{-k}\partial_{\alpha\dot{\alpha}}\Omega^k, \quad \forall k \in \mathbb{Z}. \tag{2.221}$$

Contracting both sides with $\Omega^{-2}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}$ gives

$$\hat{\nabla}^{\beta\dot{\beta}}\hat{\phi}_{\beta\dots\gamma} = \Omega^{-3}\partial^{\beta\dot{\beta}}\phi_{\beta\dots\gamma}. \tag{2.222}$$

Hence, if Eq. (2.219) is satisfied in Minkowski space-time, then it will be satisfied in any conformally flat space-time, and a similar argument holds true for the positive helicity z.r.m. equations. For the scalar case, the massless fields obey the conformally-coupled wave equation, given by

$$\left(\Box + \frac{R}{6}\right)\Phi = 0, \tag{2.223}$$

where R is the Ricci scalar of the conformally rescaled metric.

The Penrose Transform

The key observation is that these z.r.m. equations are conformally invariant, and conformal invariance is naturally encoded in twistor space. This suggests that solutions to these equations should have a natural twistorial description. Given a point $x \in M_{\mathbb{C}}$, which corresponds to a line $X \cong \mathbb{CP}^1$ in $\mathbb{PT}$, a z.r.m. field at x should arise by integrating some twistor data over X. For a negative helicity field of spin s, one is naturally led to the integral formula

$$\phi_{\alpha_1\dots\alpha_{2s}}(x) = \int_X \langle \lambda\, \mathrm{d}\lambda\rangle \wedge \lambda_{\alpha_1}\cdots\lambda_{\alpha_{2s}}\, f(Z)\big|_X, \tag{2.224}$$

where $\langle \mathrm{d}\lambda\, \lambda\rangle$ is the natural holomorphic measure on $\mathbb{CP}^1$ of projective weight $+2$. For this integral to be well-defined, the integrand must have homogeneity zero, which forces f to be a $(0,1)$-form on $\mathbb{PT}$ of projective weight $-2s-2$, that is,

$$f \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-2s-2)). \tag{2.225}$$

One can verify that $\phi_{\alpha_1\dots\alpha_{2s}}$ satisfies the z.r.m. equation (2.219) if and only if $\bar{\partial}f = 0$ on $\mathbb{PT}$. Moreover, adding a $\bar{\partial}$-exact form $f \mapsto f + \bar{\partial}g$ leaves the space-time field unchanged. The relevant space of twistor representatives is therefore the Dolbeault cohomology group $H^{0,1}(\mathbb{PT}, \mathcal{O}(-2s-2))$, where we quotient $\bar{\partial}$-closed forms by $\bar{\partial}$-exact ones. The full result, which also holds in the positive helicity case, is the Penrose transform:

$$\{\text{helicity } h \text{ z.r.m. fields on } M_{\mathbb{C}}\} \;\cong\; H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2))\,, \tag{2.226}$$

for h any integer or half-integer. This isomorphism has a direct bearing on the material in Chapter 3. The anti-self-dual Yang–Mills equation is the condition that the curvature $F_{\alpha\beta}$ of a gauge connection vanishes, which is precisely the negative helicity z.r.m. equation promoted to the non-Abelian setting. Its twistor description is therefore the natural non-Abelian generalisation of (2.226): rather than a cohomology class, the relevant data is a holomorphic vector bundle $E \to \mathbb{PT}$ which is trivial on every line $X \cong \mathbb{CP}^1$ corresponding to a point of space-time. This is the Penrose–Ward transform, and the condition of holomorphic triviality on lines is precisely the integrability condition for the Lax pair for all values of the spectral parameter ζ. It is this geometric fact that motivates studying holomorphic Chern–Simons theory on twistor space in Chapter 3: its equations of motion $\bar{\mathcal{F}}(\bar{\mathcal{A}}) = 0$ are exactly the condition that $\bar{\mathcal{A}}$ defines a holomorphic structure on a bundle over $\mathbb{PT}$, which via the Penrose–Ward transform encodes an ASD connection on space-time.

3. From holomorphic to 4d Chern–Simons theory

The material in this chapter closely follows Bittleston and Skinner [3]. We now turn to the central construction of this report: holomorphic Chern–Simons theory on twistor space. Having established the necessary background on integrability, Chern–Simons theory, and twistor geometry in Chapter 2, we will now consider how these structures fit together into a unified framework. Specifically, we will demonstrate that holomorphic Chern–Simons theory on twistor space simultaneously encodes the four-dimensional Wess–Zumino–Witten model and four-dimensional Chern–Simons theory, with the two-dimensional principal chiral model appearing as a common dimensional reduction. The spectral parameter of the integrable structure emerges naturally from the $\mathbb{CP}^1$ fibration of twistor space, making the integrability of these models manifest from a unified geometric perspective.

3.1 Holomorphic Chern–Simons theory on twistor space

Working on the Euclidean twistor space $\mathbb{PT}$ introduced in Section 2.4.2, with its identification $\mathbb{PT} \cong \mathbb{R}^4 \times \mathbb{CP}^1$, we now define holomorphic Chern–Simons theory (HCS). The action of HCS comes with a partial connection $\bar{\partial} + \bar{\mathcal{A}}$ on a principal G-bundle and is integrated over a 3d complex manifold $\mathcal{W}$, where G is a simple, complex group.

$$S_{\mathrm{HCS}}[\bar{\mathcal{A}}] = \frac{1}{2\pi i}\int_{\mathcal{W}} \Omega \wedge \mathrm{HCS}(\bar{\mathcal{A}}), \quad \mathrm{HCS}(\bar{\mathcal{A}}) = \mathrm{tr}\left(\bar{\mathcal{A}} \wedge \bar{\partial}\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right). \tag{3.1}$$

Ω here is a holomorphic (3,0)-form. We want to have $\mathcal{W} = \mathbb{PT}$ or its compactification $\mathbb{CP}^3$. D^3Z seems like a good choice for Ω, but does not work well since it has holomorphic weight 4. Costello [4] solved this issue by choosing a pair of dual twistors $\alpha, \beta \in (\mathbb{C}^4)^*$ and setting

$$\Omega = \frac{D^3Z}{(Z\cdot\alpha)^2(Z\cdot\beta)^2}. \tag{3.2}$$

This is now weightless, but is now meromorphic instead of holomorphic, since the planes where $Z\cdot\alpha = 0$ or $Z\cdot\beta = 0$ make this singular. They each define a $\mathbb{CP}^2 \subset \mathbb{CP}^3$ and intersect on a $\mathbb{CP}^1$, and removing this $\mathbb{CP}^1$ from $\mathbb{CP}^3$ leaves us with $\mathbb{PT}$. The dual twistors are given by $\alpha^A = (0^{\dot{\alpha}}, a_\alpha)$ and $\beta^A = (0^{\dot{\alpha}}, b_\alpha)$ for some left handed Weyl spinors a_α and b_α. These can be chosen to be normalised as $\langle b\ a\rangle = \epsilon_{\alpha\beta}a^\alpha b^\beta = 1$. To eliminate issues arising from the poles of Ω, we require that $\bar{\mathcal{A}}$ take the form

$$\bar{\mathcal{A}} = (Z\cdot\alpha)(Z\cdot\beta)\varphi \tag{3.3}$$

for some $\varphi \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g}\otimes\mathcal{O}(-2))$, where $\mathfrak{g}$ is the Lie algebra of the gauge group G. This is done to ensure $\bar{\mathcal{A}}$ is weightless. Also, since $\bar{\mathcal{A}}$ vanishes on $\{\alpha\cdot Z = 0\}\cup\{\beta\cdot Z = 0\}$, this must be preserved under gauge transformations $\bar{\mathcal{A}} \mapsto \bar{\mathcal{A}} + \bar{\partial}\varepsilon + [\bar{\mathcal{A}}, \varepsilon]$, so ε must take the form

$$\varepsilon = (Z\cdot\alpha)(Z\cdot\beta)\psi \tag{3.4}$$

for some $\psi \in \Omega^0(\mathbb{PT}, \mathfrak{g}\otimes\mathcal{O}(-2))$. The classical equations of motion of HCS are

$$\bar{\mathcal{F}}(\bar{\mathcal{A}}) = \bar{\partial}\bar{\mathcal{A}} + \bar{\mathcal{A}}\wedge\bar{\mathcal{A}} = 0, \tag{3.5}$$

which can be seen by simply varying the action with respect to $\bar{\mathcal{A}}$. Using the frame in (2.194), we can decompose into components along the base and fibres of $\mathbb{PT} = (\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1)$:

$$\bar{\mathcal{A}} = \bar{\mathcal{A}}_0 \bar{e}^0 + \hat{\mathcal{A}}_{\dot{\alpha}} \hat{e}^{\dot{\alpha}}, \tag{3.6}$$

$$\bar{\mathcal{F}}_{0\dot{\alpha}}(\bar{\mathcal{A}}) = \bar{\partial}_0 \hat{\mathcal{A}}_{\dot{\alpha}} - \hat{\partial}_{\dot{\alpha}} \bar{\mathcal{A}}_0 + [\bar{\mathcal{A}}_0, \hat{\mathcal{A}}_{\dot{\alpha}}] = 0, \tag{3.7}$$

$$\bar{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}(\bar{\mathcal{A}}) = \hat{\partial}_{\dot{\alpha}} \hat{\mathcal{A}}_{\dot{\beta}} - \hat{\partial}_{\dot{\beta}} \hat{\mathcal{A}}_{\dot{\alpha}} + [\hat{\mathcal{A}}_{\dot{\alpha}}, \hat{\mathcal{A}}_{\dot{\beta}}] = 0. \tag{3.8}$$

3.2 Effective description as a 4d Wess–Zumino–Witten model

In order to determine the space-time action of the Chern–Simons theory, we follow the approach of Bittleston and Skinner in [3], originally due to Costello [4]. The map Π is defined by

$$\Pi : \mathbb{PT} \to \mathbb{R}^4, \tag{3.9}$$

$$\left(\mu^{\dot{\alpha}}, \lambda_\alpha\right) \mapsto x^{\alpha\dot{\alpha}}, \tag{3.10}$$

where $x^{\alpha\dot{\alpha}}$ is given by Eq. (2.188). Using this and the map ι_x defined in Eq. (2.189), we can pull back the partial gauge field $\bar{\mathcal{A}}$ via the natural inclusion

$$\mathbb{CP}^1_x \overset{\iota_x}{\hookrightarrow} \mathbb{PT} \overset{\Pi}{\to} \mathbb{R}^4, \tag{3.11}$$

where the image of this map is a set of $\mathbb{R}^4$. We can define $\bar{\mathcal{A}}_x \equiv \iota_x^*(\bar{\mathcal{A}}) = \iota_x^*(\bar{\mathcal{A}}_0 \bar{e}^0)$. Any complex bundle on $\mathbb{CP}^1$ that is topologically trivial is generically also holomorphically trivial by Grothendieck's theorem, which states that every holomorphic vector bundle on $\mathbb{CP}^1$ splits as a direct sum of line bundles $\mathcal{O}(k)$, with topological triviality forcing $k = 0$ for each factor. It follows that the restriction of the partial connection $\bar{\partial} + \bar{\mathcal{A}}$ to each fibre $\mathbb{CP}^1_x$ can be gauged to the trivial connection, so

$$\bar{\mathcal{A}}_x = \hat{\sigma}^{-1} \bar{\partial}_{\mathbb{CP}^1_x} \hat{\sigma} \tag{3.12}$$

for some frame field $\hat{\sigma} : \mathbb{PT} \to G$. A finite gauge transformation g acts on this by $\hat{\sigma} \to \hat{\sigma} g^{-1}$. The boundary conditions on g imposed by the holomorphic Chern–Simons theory,

$$a^\alpha \partial_{\alpha\dot{\alpha}} g\big|_{\lambda \sim a} = 0, \qquad b^\alpha \partial_{\alpha\dot{\alpha}} g\big|_{\lambda \sim b} = 0, \tag{3.13}$$

prevent us from setting $\hat{\sigma} = \mathrm{id}$ everywhere, since not all gauge transformations are available to us. Left multiplication by h is also permitted as long as $\bar{\partial}_{\mathbb{CP}^1_x} h = 0$, i.e. h is independent of λ and $h : \mathbb{R}^4 \to G$. h is naturally interpreted as a gauge transformation of the effective description on $\mathbb{R}^4$. The redundancy of $\hat{\sigma}$ under left translations is eliminated by fixing $\hat{\sigma}|_{\lambda \sim b} = \mathrm{id}$, and $\hat{\sigma}|_{\lambda \sim a} = \sigma$. σ is then the holomorphic Wilson line [20, 21] from α to β

$$\mathcal{W}_{\alpha \to \beta} = P \exp\left(-\frac{1}{2\pi i} \int_{\mathbb{CP}^1_x} \frac{\langle \lambda \, \mathrm{d}\lambda \rangle}{\langle a \, \lambda \rangle \langle \lambda \, b \rangle} \wedge \bar{\mathcal{A}}_x \right) = \hat{\sigma}^{-1}|_{\lambda \sim b} \hat{\sigma}|_{\lambda \sim a} = \sigma. \tag{3.14}$$

A specific gauge is chosen by picking a specific $\hat{\sigma}$ which equals σ near a and is the identity near b. It is also useful to choose it to be invariant under the natural U_1 action on $\mathbb{CP}^1_x$ preserving a and b. It is then natural to say

$$\bar{\partial} + \bar{\mathcal{A}} = \hat{\sigma}^{-1} (\bar{\partial} + \bar{\mathcal{A}}') \hat{\sigma}, \tag{3.15}$$

where $\bar{\mathcal{A}}'$ has the condition that $\iota_x^* \bar{\mathcal{A}}' = 0$, or $\bar{\mathcal{A}}' = \hat{\mathcal{A}}_{\dot{\alpha}} \hat{e}^{\dot{\alpha}}$. One can think of $\bar{\mathcal{A}}'$ as being in a "gauge" where $\bar{\mathcal{A}}_x = 0$. The next step is to solve for $\bar{\mathcal{A}}'$ in terms of $\hat{\sigma}$ by imposing the classical equations of motion along the fibres of the fibration Π subject to the boundary conditions specified above. Using the equations of motion, we find

$$\bar{F}_{0\dot{\alpha}}(\bar{\mathcal{A}}) = \hat{\sigma}^{-1} \bar{F}_{0\dot{\alpha}}(\bar{\mathcal{A}}') \hat{\sigma} = \hat{\sigma}^{-1} \left(\bar{\partial}_0 \hat{\mathcal{A}}'_{\dot{\alpha}} \right) \hat{\sigma} = 0. \tag{3.16}$$

It follows that the components $\hat{\mathcal{A}}'_{\dot{\alpha}}$ must be holomorphic functions of the homogeneous fibre coordinate λ and hence linear, as they take values in $\mathcal{O}(1)$. Imposing the chosen boundary conditions uniquely determines the solution to be

$$\hat{\mathcal{A}}'_{\dot{\alpha}} = \lambda^\alpha A_{\alpha\dot{\alpha}} = -\langle \lambda \, b \rangle a^\beta \partial_{\dot{\alpha}\beta} \sigma \, \sigma^{-1}. \tag{3.17}$$

In particular,

$$A_{\alpha\dot\alpha} = -\,b_\alpha a^\beta \partial_{\beta\dot\alpha}\sigma\,\sigma^{-1} \tag{3.18}$$

is independent of the fibre coordinate λ and coincides with the standard expression for an anti-self-dual gauge field written in terms of the Yang matrix σ in the gauge $b^\alpha A_{\alpha\dot\alpha} = 0$, as derived in [3]. Having fully determined the λ-dependence of $\bar{\mathcal{A}}$, we may now explicitly perform the integrals over the twistor fibres in order to reduce the holomorphic Chern–Simons action $S_{\mathrm{HCS}}[\bar{\mathcal{A}}]$ to an effective action on $\mathbb{R}^4$. Following [3], we omit the details of this fibre integration and simply state the resulting four-dimensional action:

$$S_{\mathrm{WZW}_4}[\sigma] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\big(\mathcal{J}\wedge\star_4\mathcal{J}\big) + \frac{1}{3}\int_{\mathbb{R}^4\times[0,1]} \mu_{a,b}\wedge\mathrm{tr}\big(\tilde{\mathcal{J}}^{\,3}\big), \tag{3.19}$$

where

$$\mathcal{J} = -\,\mathrm{d}\sigma\,\sigma^{-1}, \qquad \tilde{\mathcal{J}} = -\,\tilde{\mathrm{d}}\tilde\sigma\,\tilde\sigma^{-1}. \tag{3.20}$$

We adopt the notation $\mathcal{J}$ here to avoid confusion with Yang's matrix $J = \tilde h^{-1}h$ introduced in Section 2.2.4. We note that this is an unfortunate clash of conventions in the literature: Mason and Woodhouse [1] use J for Yang's matrix, whereas Bittleston and Skinner [3] use σ, with J reserved for the current $-\mathrm{d}\sigma\,\sigma^{-1}$. Despite the different notation, σ here plays the same role as Yang's matrix J in Section 2.2.4. Here $\tilde\sigma$ denotes a smooth homotopy from σ to the identity, and $\tilde d$ is the exterior derivative on $\mathbb{R}^4\times[0,1]$. We have also introduced the two-form

$$\mu_{a,b} = \mathrm{d}^2x^{\alpha\beta}\,a_\alpha b_\beta, \tag{3.21}$$

which encodes the choice of complex structure on $\mathbb{R}^4$ determined by the spinors a and b. The first term is a standard kinetic term for the Yang matrix σ, while the second is a four-dimensional analogue of the Wess–Zumino term familiar from two-dimensional WZW models. This action was originally proposed by Donaldson [6]. We refer to it as the four-dimensional Wess–Zumino–Witten model (WZW_4). Varying the action with respect to σ yields the equation of motion

$$\epsilon^{\dot\alpha\dot\beta}\,a^\alpha b^\beta\,\partial_{\beta\dot\beta}\big(\partial_{\alpha\dot\alpha}\sigma\,\sigma^{-1}\big) = 0, \tag{3.22}$$

which one may recognise as Yang's equation, first derived in Section 2.2.4 in the context of the ASDYM equations, now reappearing as the equation of motion of the WZW_4. Since the ASDYM equation and Yang's equation share the same solution space, the WZW_4 model and the ASDYM equation are classically equivalent on-shell. The dimensional reduction chain from holomorphic Chern–Simons theory therefore simultaneously encodes anti-self-dual Yang–Mills theory through this shared equation of motion. This is consistent with the connection between the actions discussed in Section 2.2.5 and the effective description derived here from holomorphic Chern–Simons theory. We now specialise to the choice

$$b = \hat a, \tag{3.23}$$

in which case the two-form $\mu_{a,b} = \mu_{a,\hat a}$ is proportional to the Kähler form on $\mathbb{R}^4$ associated with the complex structure defined by a. Although this restriction is not strictly required at this stage, it becomes essential if one wishes to obtain actions for anti-self-dual connections valued in compact real forms of the gauge group G. We therefore adopt this choice. From the twistor-space construction, several structural features of the four-dimensional Wess–Zumino–Witten model become immediately apparent. The selection of the spinor a together with its conjugate $\hat a$ singles out a preferred complex structure on $\mathbb{R}^4$, reducing the left-handed rotation group $\mathrm{SU}(2)$ to a $\mathrm{U}(1)$ subgroup while leaving the right-handed factor intact, giving a symmetry group

$$\mathrm{U}(1)\times\mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{U}(2), \tag{3.24}$$

which is precisely the subgroup of $\mathrm{SO}(4)$ preserving a Kähler structure on $\mathbb{R}^4$. Furthermore, the presence of the infinity twistor

$$I^{AB} = \epsilon^{ABCD}A_C B_D \tag{3.25}$$

breaks conformal invariance down to the Poincaré group, so the WZW_4 model is not conformally invariant even at the classical level. For this choice of Ω, the theory also admits an enlarged class of gauge transformations providing a four-dimensional analogue of loop-group symmetries, and may be further reduced to the Kähler Chern–Simons theory of Nair and Schiff [22]. Having obtained WZW_4 as an effective description, we now ask what lower-dimensional integrable models it contains. We implement a symmetry reduction by a two-dimensional abelian group of translations, which will yield the principal chiral model with a Wess–Zumino term.

3.3 Symmetry reduction to a principal chiral model with Wess–Zumino–Witten term

We now implement a symmetry reduction of $\mathbb{R}^4$ by a two-dimensional abelian group of translations, which we denote by H. Without loss of generality, we may choose generators of H to be the complex null vector fields

$$X = \hat{\kappa}^{\dot{\alpha}}\gamma^{\alpha}\partial_{\alpha\dot{\alpha}}, \qquad \bar{X} = \kappa^{\dot{\alpha}}\hat{\gamma}^{\alpha}\partial_{\alpha\dot{\alpha}}. \tag{3.26}$$

The spinors are normalised such that $\|\kappa\|^2 = \|\gamma\|^2 = 1$, so that the vectors satisfy $\delta(X, \bar{X}) = 1$, where δ is the flat metric on $\mathbb{R}^4$. Given the spinor dyads (2D bases) $\{\kappa, \hat{\kappa}\}$ and $\{\gamma, \hat{\gamma}\}$, we may introduce complex double-null coordinates on $\mathbb{R}^4$ defined by

$$\begin{aligned} z &= x^{\alpha\dot{\alpha}}\kappa_{\dot{\alpha}}\gamma_{\alpha}, & w &= -x^{\alpha\dot{\alpha}}\hat{\kappa}_{\dot{\alpha}}\gamma_{\alpha}, \\ \bar{w} &= x^{\alpha\dot{\alpha}}\kappa_{\dot{\alpha}}\hat{\gamma}_{\alpha}, & \bar{z} &= x^{\alpha\dot{\alpha}}\hat{\kappa}_{\dot{\alpha}}\hat{\gamma}_{\alpha}. \end{aligned} \tag{3.27}$$

In these coordinates, the vector fields generating the symmetry reduce to

$$X = \partial_z, \qquad \bar{X} = \partial_{\bar{z}}. \tag{3.28}$$

The symmetry reduction is now applied directly to the four-dimensional Wess–Zumino–Witten action from before. To this end, we impose invariance of the Yang matrix σ under the action of H,

$$\mathcal{L}_X\sigma = \mathcal{L}_{\bar{X}}\sigma = 0. \tag{3.29}$$

In order to define these Lie derivatives, one must choose a lift of the action of H to the principal G-bundle. Working in a frame which is invariant under this lifted action ensures that the Lie derivatives act on σ as ordinary derivatives. However, this choice restricts the available gauge transformations to those satisfying $\mathcal{L}_X g = \mathcal{L}_{\bar{X}} g = 0$. As a result, the two-dimensional integrable model obtained after reduction depends on the gauge in which the anti-self-dual Yang–Mills equations are expressed, and therefore on the choice of the spinors a and $\hat{a}$. Under the condition (3.29), the field σ is independent of the coordinates z and $\bar{z}$. The integrals over these directions in the action therefore diverge, reflecting the non-compactness of the translation group $H \cong \mathbb{R}^2$. While one could regulate this divergence by compactifying $\mathbb{R}^4$ to $\mathbb{R}^2 \times T^2$ and replacing H by T^2, we avoid the associated global issues by simply removing the divergent factor $\mathrm{Vol}(H)$. Operationally, this is achieved by contracting the bivector $X \wedge \bar{X}$ into the four-dimensional Lagrangian density of the WZW_4 model, viewed as a top-degree form. This procedure saturates the form components along the symmetry directions. Assuming invariance under H, the resulting two-form descends naturally to the quotient space. It is convenient to identify this quotient with the transverse slice

$$\Sigma = \{z = \bar{z} = 0\} \subset \mathbb{R}^4. \tag{3.30}$$

Accordingly, the pushforward to the quotient may be replaced by the pullback along the inclusion map $\iota_\Sigma : \Sigma \hookrightarrow \mathbb{R}^4$. A straightforward computation yields

$$\begin{aligned} (X \wedge \bar{X}) \lrcorner \operatorname{tr}(\mathcal{J} \wedge \star_4 \mathcal{J}) &= \operatorname{tr}(\mathcal{J} \wedge \star_2 \mathcal{J}), \\ (X \wedge \bar{X}) \lrcorner \big(\mu_{a,\hat{a}} \wedge \operatorname{tr}(\tilde{\mathcal{J}}^{\,3})\big) &= k \operatorname{tr}(\tilde{\mathcal{J}}^{\,3}), \end{aligned} \tag{3.31}$$

where the constant k is given by

$$k = (X \wedge \bar{X}) \lrcorner \mu_{a,\hat{a}} = \langle a\,\gamma\rangle\langle\hat{a}\,\hat{\gamma}\rangle + \langle a\,\hat{\gamma}\rangle\langle\hat{a}\,\gamma\rangle. \tag{3.32}$$

The symmetry reduction of the WZW_4 model by H therefore yields the two-dimensional principal chiral model with a Wess–Zumino term,

$$S_{\mathrm{PCM}}[\sigma] = \frac{1}{2}\int_\Sigma \operatorname{tr}(\mathcal{J} \wedge \star_2 \mathcal{J}) + \frac{k}{3}\int_{\Sigma\times[0,1]} \operatorname{tr}(\tilde{\mathcal{J}}^{\,3}), \tag{3.33}$$

where $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are defined as before. Two special cases are of particular interest. If the spinor a is proportional to either γ or $\hat{\gamma}$, then $k = \mp 1$ and the action reduces to that of the conventional two-dimensional Wess–Zumino–Witten model. On the other hand, when $k = 0$, which occurs when a lies on the equator of the Riemann sphere determined by γ, the Wess–Zumino term vanishes and one recovers the principal chiral model without a topological term. Different values of k therefore correspond to distinct integrable systems. For example, when σ takes values in a compact real form of $\mathrm{SU}(n)$, the principal chiral model without a Wess–Zumino term admits classical solutions extending smoothly over S^2, whereas this is not the case for the WZW_2 model.

3.4 Symmetry reduction to 4d Chern–Simons theory

Although both anti–self–dual Yang–Mills theory and the principal chiral model are integrable, their formulations on $\mathbb{R}^4$ and $\mathbb{R}^2$ do not make the associated spectral parameter manifest. From the twistor viewpoint this parameter has a geometric origin, appearing as the homogeneous coordinate λ on the $\mathbb{CP}^1$ fibres of twistor space. In this section we exploit this fact by implementing the symmetry reduction with respect to the translation group H directly at the level of holomorphic Chern–Simons theory on twistor space. The resulting theory coincides with the four–dimensional Chern–Simons description of the principal chiral model. We begin by extending the action of H from space–time to twistor space $\mathbb{PT}$. Equivalently, we lift the generators X and $\bar{X}$ to vector fields on $\mathbb{PT}$ that project to the original translations under the fibration $\Pi : \mathbb{PT} \to \mathbb{R}^4$ and preserve the complex structure. For translations in flat Euclidean space this lift is straightforward and is given by

$$\mathcal{X} = \hat{\kappa}^{\dot{\alpha}}\hat{\gamma}^{\alpha}\partial_{\alpha\dot{\alpha}} = \partial_z, \qquad \bar{\mathcal{X}} = \kappa^{\dot{\alpha}}\gamma^{\alpha}\partial_{\alpha\dot{\alpha}} = \partial_{\bar{z}}. \tag{3.34}$$

Using the identification of the smooth twistor space with $\mathbb{PS}^+ \cong \mathbb{R}^4 \times \mathbb{CP}^1$, the quotient by H may be represented by the submanifold

$$\mathcal{V} = \{z = \bar{z} = 0\} \subset \mathbb{PT}, \tag{3.35}$$

which intersects each orbit of H exactly once. We denote the natural embedding by $\iota_{\mathcal{V}} : \mathcal{V} \hookrightarrow \mathbb{PT}$ and continue to label the remaining coordinates on $\mathcal{V}$ by $(w, \bar{w}, \lambda_\alpha)$. The reduction is implemented by imposing invariance of the twistor gauge field $\bar{\mathcal{A}}$ under the lifted translations,

$$\mathcal{L}_{\mathcal{X}}\bar{\mathcal{A}} = \mathcal{L}_{\bar{\mathcal{X}}}\bar{\mathcal{A}} = 0. \tag{3.36}$$

As before, the Lie derivative is defined using a lift of the H–action to the principal G–bundle. Working in a frame invariant under this action ensures that the Lie derivative acts on $\bar{\mathcal{A}}$ as on an ordinary differential form. The remaining gauge freedom consists of transformations that are constant along the H–orbits. Since the holomorphic Chern–Simons Lagrangian is invariant under H, the integral over the coordinates $(z, \bar{z})$ produces a divergent factor corresponding to the volume of H. Rather than compactifying these directions, we remove this divergence by contracting the bivector $\mathcal{X} \wedge \bar{\mathcal{X}}$ into the six–form $\Omega \wedge \mathrm{HCS}(\bar{A})$ and then restricting to the slice $\mathcal{V}$. Explicitly, one finds

$$\iota_{\mathcal{V}}^*\big((\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner (\Omega \wedge \mathrm{HCS}(\bar{\mathcal{A}}))\big) = \frac{\langle\lambda\gamma\rangle\langle\lambda\hat{\gamma}\rangle\langle\lambda\,\mathrm{d}\lambda\rangle}{\langle\lambda a\rangle^2\langle\lambda\hat{a}\rangle^2} \wedge \mathrm{PHCS}(A'), \tag{3.37}$$

where the reduced gauge field A' is defined by

$$A' = \iota_{\mathcal{V}}^*\bar{\mathcal{A}} - \frac{\langle\lambda\hat{\gamma}\rangle}{\langle\lambda\gamma\rangle}\,dw\,\iota_{\mathcal{V}}^*(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) + \frac{\langle\lambda\gamma\rangle}{\langle\lambda\hat{\gamma}\rangle}\,d\bar{w}\,\iota_{\mathcal{V}}^*(\mathcal{X} \lrcorner \bar{\mathcal{A}}), \tag{3.38}$$

and the partially holomorphic Chern–Simons form is

$$\mathrm{PHCS}(A') = \mathrm{tr}\left(A' \wedge \mathrm{d}'A' + \frac{2}{3}A' \wedge A' \wedge A'\right). \tag{3.39}$$

Here the differential operator

$$\mathrm{d}' = \bar{\partial}_{\mathbb{CP}^1} + \mathrm{d}_{\mathbb{R}^2} \tag{3.40}$$

defines a simple partially holomorphic structure on $\mathcal{V} \cong \mathbb{R}^2 \times \mathbb{CP}^1$. To show Eq. (3.37), we first notice that

$$\iota_{\mathcal{V}}^*\big((\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner (\Omega \wedge \mathrm{HCS}(\bar{\mathcal{A}}))\big) = \iota_{\mathcal{V}}^*\big((\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner (\Omega \wedge \mathrm{CS}(\bar{\mathcal{A}}))\big)\,. \tag{3.41}$$

This is because $\Omega \wedge \mathrm{HCS}(\bar{\mathcal{A}})$ is already a (3,3)-form, so the additional legs from replacing $\bar{\partial}$ with d do not contribute. We then need to show

$$\iota_{\mathcal{V}}^*\big((\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner (\Omega \wedge \mathrm{CS}(\bar{\mathcal{A}}))\big) = \frac{\langle\lambda\gamma\rangle\langle\lambda\hat{\gamma}\rangle\langle\lambda\,\mathrm{d}\lambda\rangle}{\langle\lambda a\rangle^2\langle\lambda\hat{a}\rangle^2} \;\wedge\; \mathrm{CS}(A'). \tag{3.42}$$

We use the standard convention

$$(\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner \omega = 2\bar{\mathcal{X}} \lrcorner (\mathcal{X} \lrcorner \omega). \tag{3.43}$$

For any forms ω, η,

$$\mathcal{X} \lrcorner (\omega \wedge \eta) = (\mathcal{X} \lrcorner \omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge (\mathcal{X} \lrcorner \eta). \tag{3.44}$$

Thus

$$(\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner (\Omega \wedge \mathrm{CS}(\bar{\mathcal{A}})) = 2\bar{\mathcal{X}} \lrcorner (\mathcal{X} \lrcorner (\Omega \wedge \mathrm{CS}(\bar{\mathcal{A}}))). \tag{3.45}$$

We now use this to compute $(\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner (\Omega \wedge \mathrm{CS}(\bar{\mathcal{A}}))$ explicitly. Expanding the first contraction:

$$\mathcal{X} \lrcorner (\Omega \wedge \mathrm{CS}(\bar{\mathcal{A}})) = (\mathcal{X} \lrcorner \Omega) \wedge \mathrm{CS}(\bar{\mathcal{A}}) - \Omega \wedge (\mathcal{X} \lrcorner \mathrm{CS}(\bar{\mathcal{A}})), \tag{3.46}$$

since $\deg \Omega = 3$. Applying $\bar{\mathcal{X}} \lrcorner$ gives

$$\begin{aligned}(\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner (\Omega \wedge \mathrm{CS}(\bar{\mathcal{A}})) = 2((\bar{\mathcal{X}} \lrcorner (\mathcal{X} \lrcorner \Omega)) \wedge \mathrm{CS}(\bar{\mathcal{A}}) + (\mathcal{X} \lrcorner \Omega) \wedge (\bar{\mathcal{X}} \lrcorner \mathrm{CS}(\bar{\mathcal{A}})) \\ - (\bar{\mathcal{X}} \lrcorner \Omega) \wedge (\mathcal{X} \lrcorner \mathrm{CS}(\bar{\mathcal{A}})) + \Omega \wedge (\bar{\mathcal{X}} \lrcorner (\mathcal{X} \lrcorner \mathrm{CS}(\bar{\mathcal{A}})))).\end{aligned} \tag{3.47}$$

Using the explicit form

$$\Omega = \frac{\langle \lambda \, \mathrm{d}\lambda \rangle \wedge \epsilon_{\dot{\beta}\dot{\alpha}} \lambda_\alpha \lambda_\beta \mathrm{d}x^{\alpha\dot{\alpha}} \wedge \mathrm{d}x^{\beta\dot{\beta}}}{4\langle \lambda a \rangle^2 \langle \lambda \hat{a} \rangle^2}, \tag{3.48}$$

and

$$\mathcal{X} = \partial_z, \qquad \bar{\mathcal{X}} = \partial_{\bar{z}}, \tag{3.49}$$

we obtain

$$\begin{aligned}\mathcal{X} \lrcorner \Omega &= \frac{\langle \lambda \, \mathrm{d}\lambda \rangle \langle \hat{\gamma} \lambda \rangle \wedge \epsilon_{\dot{\beta}\dot{\alpha}} \lambda_\alpha \hat{\kappa}^{\dot{\beta}} \mathrm{d}x^{\alpha\dot{\alpha}}}{2\langle \lambda a \rangle^2 \langle \lambda \hat{a} \rangle^2}, \\ \bar{\mathcal{X}} \lrcorner \Omega &= \frac{\langle \lambda \, \mathrm{d}\lambda \rangle \langle \gamma \lambda \rangle \wedge \epsilon_{\dot{\beta}\dot{\alpha}} \lambda_\alpha \kappa^{\dot{\beta}} \mathrm{d}x^{\alpha\dot{\alpha}}}{2\langle \lambda a \rangle^2 \langle \lambda \hat{a} \rangle^2}, \\ (\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner \Omega &= \frac{\langle \lambda \, \mathrm{d}\lambda \rangle \langle \lambda \hat{\gamma} \rangle \langle \lambda \gamma \rangle}{2\langle \lambda a \rangle^2 \langle \lambda \hat{a} \rangle^2}.\end{aligned} \tag{3.50}$$

Having computed the contractions into Ω, we now turn to the contractions of $\mathcal{X}$ and $\bar{\mathcal{X}}$ into $\mathrm{CS}(\bar{\mathcal{A}})$.

$$\begin{aligned}\mathcal{X} \lrcorner \mathrm{CS}(\bar{\mathcal{A}}) &= \mathrm{tr}\left((\mathcal{X} \lrcorner \bar{\mathcal{A}}) \mathrm{d}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \mathrm{d}(\mathcal{X} \lrcorner \bar{\mathcal{A}}) + 2(\mathcal{X} \lrcorner \bar{\mathcal{A}}) \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right), \\ \bar{\mathcal{X}} \lrcorner \mathrm{CS}(\bar{\mathcal{A}}) &= \mathrm{tr}\left((\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \mathrm{d}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \mathrm{d}(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) + 2(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right), \\ \bar{\mathcal{X}} \lrcorner (\mathcal{X} \lrcorner \mathrm{CS}(\bar{\mathcal{A}})) &= \mathrm{tr}\Big((\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \mathrm{d}(\mathcal{X} \lrcorner \bar{\mathcal{A}}) - (\mathcal{X} \lrcorner \bar{\mathcal{A}}) \wedge \mathrm{d}(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \\ &\qquad + 2(\mathcal{X} \lrcorner \bar{\mathcal{A}})(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \bar{\mathcal{A}} - 2(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})(\mathcal{X} \lrcorner \bar{\mathcal{A}}) \bar{\mathcal{A}}\Big).\end{aligned} \tag{3.51}$$

The first term of the full contraction is clearly the following:

$$2((\mathcal{X} \wedge \bar{\mathcal{X}}) \lrcorner \Omega) \wedge \mathrm{CS}(\bar{\mathcal{A}}) = \frac{\langle \lambda \, \mathrm{d}\lambda \rangle \langle \lambda \hat{\gamma} \rangle \langle \lambda \gamma \rangle}{\langle \lambda a \rangle^2 \langle \lambda \hat{a} \rangle^2} \wedge \mathrm{CS}(\bar{\mathcal{A}}). \tag{3.52}$$

The other terms are less straightforward. Substituting directly, we have

$$\begin{aligned}2(\mathcal{X} \lrcorner \Omega) \wedge (\bar{\mathcal{X}} \lrcorner \mathrm{CS}(\bar{\mathcal{A}})) = \frac{\langle \lambda \, \mathrm{d}\lambda \rangle \langle \hat{\gamma} \lambda \rangle \wedge \epsilon_{\dot{\beta}\dot{\alpha}} \lambda_\alpha \hat{\kappa}^\beta \mathrm{d}x^{\alpha\dot{\alpha}}}{\langle \lambda a \rangle^2 \langle \lambda \hat{a} \rangle^2} \\ \wedge \mathrm{tr}\left((\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \mathrm{d}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \mathrm{d}(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) + 2(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right).\end{aligned} \tag{3.53}$$

Since the $\mathrm{d}\bar{z}$ leg is forced to be in the CS form, the only non-vanishing part of $dx^{\alpha\dot{\alpha}}$ is the dw leg, which can be seen by writing $x^{\alpha\dot{\alpha}}$ as

$$x^{\alpha\dot{\alpha}} = z\hat{\kappa}^{\dot{\alpha}} \hat{\gamma}^\alpha + w\kappa^{\dot{\alpha}} \hat{\gamma}^\alpha - \bar{w}\hat{\kappa}^{\dot{\alpha}} \gamma^\alpha + \bar{z}\kappa^{\dot{\alpha}} \gamma^\alpha. \tag{3.54}$$

Since there is already a $\hat{\kappa}$ in the form, only the w term of $x^{\alpha\dot{\alpha}}$ will contribute, giving us

$$\begin{aligned}2(\mathcal{X} \lrcorner \Omega) \wedge (\bar{\mathcal{X}} \lrcorner \mathrm{CS}(\bar{\mathcal{A}})) = -\frac{\langle \lambda \, \mathrm{d}\lambda \rangle \langle \lambda \hat{\gamma} \rangle^2 \wedge \mathrm{d}w}{\langle \lambda \alpha \rangle^2 \langle \lambda \hat{\alpha} \rangle^2} \\ \wedge \mathrm{tr}\left((\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \mathrm{d}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \mathrm{d}(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) + 2(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right).\end{aligned} \tag{3.55}$$

Since $\mathrm{d}w$ is just a coordinate, we can move it inside the trace,

$$\begin{aligned} 2(\mathcal{X} \lrcorner \Omega) \wedge (\bar{\mathcal{X}} \lrcorner \mathrm{CS}(\bar{\mathcal{A}})) = &-\frac{\langle \lambda\, \mathrm{d}\lambda\rangle \langle \lambda\hat{\gamma}\rangle^2}{\langle \lambda a\rangle^2 \langle \lambda \hat{a}\rangle^2} \\ &\wedge \mathrm{tr}\Big((\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})\mathrm{d}w \wedge \mathrm{d}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \mathrm{d}(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}\, \mathrm{d}w) \\ &\qquad + 2(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})\, \mathrm{d}w \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\Big). \end{aligned} \tag{3.56}$$

Similarly, we have

$$\begin{aligned} 2(\bar{\mathcal{X}} \lrcorner \Omega) \wedge (\mathcal{X} \lrcorner \mathrm{CS}(\bar{\mathcal{A}})) = &\frac{\langle \lambda\, \mathrm{d}\lambda\rangle \langle \lambda\gamma\rangle^2}{\langle \lambda a\rangle^2 \langle \lambda \hat{a}\rangle^2} \\ &\wedge \mathrm{tr}\Big((\mathcal{X} \lrcorner \bar{\mathcal{A}})\mathrm{d}\bar{w} \wedge \mathrm{d}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \mathrm{d}(\mathcal{X} \lrcorner \bar{\mathcal{A}}\, \mathrm{d}\bar{w}) \\ &\qquad + 2(\mathcal{X} \lrcorner \bar{\mathcal{A}})\mathrm{d}\bar{w} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\Big). \end{aligned} \tag{3.57}$$

Now all that is left is to substitute in the last term

$$\begin{aligned} 2\Omega \wedge (\bar{\mathcal{X}} \lrcorner (\mathcal{X} \lrcorner \mathrm{CS}(\bar{\mathcal{A}}))) = &\frac{\langle \lambda\, \mathrm{d}\lambda\rangle \wedge \epsilon_{\dot{\beta}\dot{\alpha}} \lambda_\alpha \lambda_\beta\, \mathrm{d}x^{\alpha\dot{\alpha}} \wedge \mathrm{d}x^{\beta\dot{\beta}}}{2\langle \lambda a\rangle^2 \langle \lambda \hat{a}\rangle^2} \wedge \mathrm{tr}\Big((\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})\, \mathrm{d}(\mathcal{X} \lrcorner \bar{\mathcal{A}}) \\ &- (\mathcal{X} \lrcorner \bar{\mathcal{A}}) \wedge \mathrm{d}(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) + 2(\mathcal{X} \lrcorner \bar{\mathcal{A}})(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})\bar{\mathcal{A}} \\ &- 2(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})(\mathcal{X} \lrcorner \bar{\mathcal{A}})\bar{\mathcal{A}}\Big). \end{aligned} \tag{3.58}$$

Writing $\mathrm{d}x^{\alpha\dot{\alpha}}$ in terms of $\mathrm{d}w$ and $\mathrm{d}\bar{w}$:

$$\begin{aligned} 2\Omega \wedge (\bar{\mathcal{X}} \lrcorner (\mathcal{X} \lrcorner \mathrm{CS}(\bar{\mathcal{A}}))) = &\frac{\langle \lambda\, \mathrm{d}\lambda\rangle \langle \lambda\gamma\rangle \langle \lambda\hat{\gamma}\rangle}{\langle \lambda a\rangle^2 \langle \lambda \hat{a}\rangle^2} \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w} \\ &\wedge \mathrm{tr}\Big((\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})\mathrm{d}(\mathcal{X} \lrcorner \bar{\mathcal{A}}) - (\mathcal{X} \lrcorner \bar{\mathcal{A}})\mathrm{d}(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}) \\ &+ 2(\mathcal{X} \lrcorner \bar{\mathcal{A}})(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})\bar{\mathcal{A}} - 2(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})(\mathcal{X} \lrcorner \bar{\mathcal{A}})\bar{\mathcal{A}}\Big). \end{aligned} \tag{3.59}$$

Once more bringing $\mathrm{d}w$ and $\mathrm{d}\bar{w}$ inside the trace, we have

$$\begin{aligned} 2\Omega \wedge (\bar{\mathcal{X}} \lrcorner (\mathcal{X} \lrcorner \mathrm{CS}(\bar{\mathcal{A}}))) = &-\frac{\langle \lambda\, \mathrm{d}\lambda\rangle \langle \lambda\gamma\rangle \langle \lambda\hat{\gamma}\rangle}{\langle \lambda a\rangle^2 \langle \lambda \hat{a}\rangle^2} \\ &\wedge \Bigg[\mathrm{tr}\left((\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})\mathrm{d}w \wedge \mathrm{d}(\mathcal{X} \lrcorner \bar{\mathcal{A}}\, \mathrm{d}\bar{w}) + (\mathcal{X} \lrcorner \bar{\mathcal{A}})\mathrm{d}\bar{w} \wedge \mathrm{d}(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}}\, \mathrm{d}w)\right) \\ &+ 2\,\mathrm{tr}\Big((\mathcal{X} \lrcorner \bar{\mathcal{A}})\mathrm{d}\bar{w} \wedge (\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})\mathrm{d}w \wedge \bar{\mathcal{A}} \\ &\qquad + 2(\bar{\mathcal{X}} \lrcorner \bar{\mathcal{A}})\mathrm{d}w \wedge (\mathcal{X} \lrcorner \bar{\mathcal{A}})\mathrm{d}\bar{w} \wedge \bar{\mathcal{A}}\Big)\Bigg]. \end{aligned} \tag{3.60}$$

It remains to compute $\mathrm{CS}(A')$ in order to verify that the assembled terms match the right-hand side of Eq. (3.42). Substituting in and simplifying, we obtain the following expression:

$$\begin{aligned}
\mathrm{CS}(A') = \mathrm{CS}(\iota_{\mathcal{V}}^*\bar{\mathcal{A}}) \\
&+ \mathrm{tr}\left(\frac{\langle\lambda\gamma\rangle}{\langle\lambda\hat{\gamma}\rangle}\Big(\iota_{\mathcal{V}}^*(\mathcal{X}\lrcorner\bar{\mathcal{A}})\mathrm{d}\bar{w}\wedge\mathrm{d}(\iota_{\mathcal{V}}^*\bar{\mathcal{A}}) + \iota_{\mathcal{V}}^*\bar{\mathcal{A}}\,\mathrm{d}\bar{w}\wedge\mathrm{d}(\iota_{\mathcal{V}}^*(\mathcal{X}\lrcorner\bar{\mathcal{A}})) \right.\right. \\
&\qquad \left.\left. + 2\iota_{\mathcal{V}}^*(\mathcal{X}\lrcorner\bar{\mathcal{A}})\mathrm{d}\bar{w}\wedge\iota_{\mathcal{V}}^*\bar{\mathcal{A}}\wedge\iota_{\mathcal{V}}^*\bar{\mathcal{A}}\Big)\right) \\
&- \mathrm{tr}\left(\frac{\langle\lambda\hat{\gamma}\rangle}{\langle\lambda\gamma\rangle}\Big(\iota_{\mathcal{V}}^*(\bar{\mathcal{X}}\lrcorner\bar{\mathcal{A}})\mathrm{d}w\wedge\mathrm{d}(\iota_{\mathcal{V}}^*\bar{\mathcal{A}}) + \iota_{\mathcal{V}}^*\bar{\mathcal{A}}\,\mathrm{d}w\wedge\mathrm{d}(\iota_{\mathcal{V}}^*(\bar{\mathcal{X}}\lrcorner\bar{\mathcal{A}})) \right.\right. \\
&\qquad \left.\left. + 2\iota_{\mathcal{V}}^*(\bar{\mathcal{X}}\lrcorner\bar{\mathcal{A}})\mathrm{d}w\wedge\iota_{\mathcal{V}}^*\bar{\mathcal{A}}\wedge\iota_{\mathcal{V}}^*\bar{\mathcal{A}}\Big)\right) \\
&- \mathrm{tr}\Big(\iota_{\mathcal{V}}^*(\bar{\mathcal{X}}\lrcorner\bar{\mathcal{A}})\mathrm{d}w\wedge\mathrm{d}(\iota_{\mathcal{V}}^*(\mathcal{X}\lrcorner\bar{\mathcal{A}})\mathrm{d}\bar{w}) \\
&\qquad + \iota_{\mathcal{V}}^*(\mathcal{X}\lrcorner\bar{\mathcal{A}})\mathrm{d}\bar{w}\wedge\mathrm{d}(\iota_{\mathcal{V}}^*(\bar{\mathcal{X}}\lrcorner\bar{\mathcal{A}})\mathrm{d}w)\Big) \\
&- 2\,\mathrm{tr}\Big(\iota_{\mathcal{V}}^*(\bar{\mathcal{X}}\lrcorner\bar{\mathcal{A}})\mathrm{d}w\wedge\iota_{\mathcal{V}}^*(\mathcal{X}\lrcorner\bar{\mathcal{A}})\mathrm{d}\bar{w}\wedge\iota_{\mathcal{V}}^*\bar{\mathcal{A}} \\
&\qquad + \iota_{\mathcal{V}}^*(\mathcal{X}\lrcorner\bar{\mathcal{A}})\mathrm{d}\bar{w}\wedge\iota_{\mathcal{V}}^*(\bar{\mathcal{X}}\lrcorner\bar{\mathcal{A}})\mathrm{d}w\wedge\iota_{\mathcal{V}}^*\bar{\mathcal{A}}\Big).
\end{aligned} \tag{3.61}$$

Now putting everything together, we have

$$\iota_{\mathcal{V}}^*\big((\mathcal{X}\wedge\bar{\mathcal{X}})\lrcorner(\Omega\wedge\mathrm{CS}(\bar{\mathcal{A}}))\big) = \omega\wedge\mathrm{CS}(A') = \omega\wedge\mathrm{PHCS}(A'). \tag{3.62}$$

The meromorphic measure appearing above is

$$\omega = \frac{\langle\lambda\gamma\rangle\langle\lambda\hat{\gamma}\rangle\langle\lambda\,\mathrm{d}\lambda\rangle}{\langle\lambda a\rangle^2\langle\lambda\hat{a}\rangle^2}, \tag{3.63}$$

which exhibits simple zeros and poles determined by the choice of spinors a and γ. The fact that this measure already has a $\mathrm{d}\lambda$ leg allows us to simplify the CS form to the PHCS form. In terms of an inhomogeneous coordinate $\zeta = \langle\lambda a\rangle/\langle\lambda\hat{a}\rangle$, this measure takes the standard rational form familiar from four–dimensional Chern–Simons theory. The action on the quotient space is therefore

$$S_{\mathrm{CS}_4}[A'] = \frac{1}{2\pi i}\int_{\mathcal{V}}\omega\wedge\mathrm{PHCS}(A'). \tag{3.64}$$

For the boundary conditions implied by the reduction, this theory is equivalent to the principal chiral model on the two–dimensional surface Σ, in agreement with the results obtained by reducing the four–dimensional Wess–Zumino–Witten model directly. This establishes that holomorphic Chern–Simons theory on twistor space descends, under symmetry reduction by H, to four-dimensional Chern–Simons theory on $\mathcal{V}\cong\mathbb{R}^2\times\mathbb{CP}^1$, with the meromorphic one-form ω inheriting its poles and zeros from the twistor geometry. The spectral parameter ζ emerges naturally from the $\mathbb{CP}^1$ fibration, making the integrable structure of the principal chiral model manifest. Combined with the results of Section 3.2, this completes the chain: holomorphic Chern–Simons theory on twistor space simultaneously encodes the four-dimensional Wess–Zumino–Witten model and four-dimensional Chern–Simons theory, with the two-dimensional principal chiral model appearing as a common dimensional reduction.

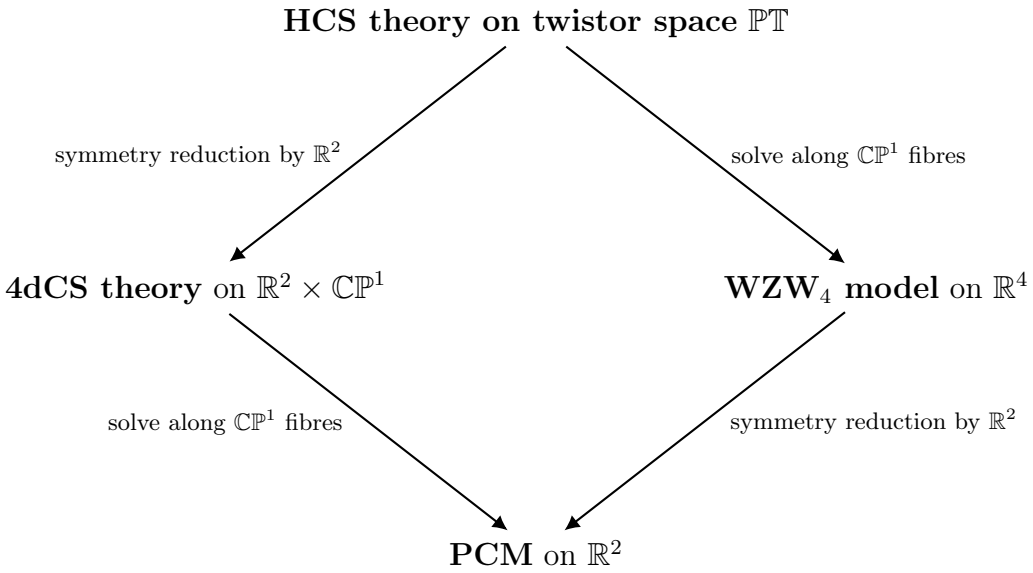


Figure 3.1: The dimensional reduction structure of holomorphic Chern–Simons theory on twistor space $\mathbb{PT}$.

4. Constructing Solutions from the Atiyah–Ward Ansatz

4.1 Patching Matrices

The Penrose–Ward transform establishes a correspondence between ASD connections on $\mathbb{E}^4$ and holomorphic vector bundles on twistor space $\mathbb{PT}$. The holomorphic bundle is described concretely by a patching matrix F, which encodes the transition function between two trivialisations of the bundle on the overlap of two patches U_+ and U_- covering $\mathbb{PT}$. More precisely, given an ASD connection D with fundamental solutions f and $\tilde{f}$ on U_+ and U_- respectively, the patching matrix is defined on the overlap $U_+ \cap U_-$ by

$$f = \tilde{f}F, \tag{4.1}$$

where F is the pullback by the twistor fibration of a holomorphic function on $U_+ \cap U_-$. A fundamental solution is an $n \times n$ matrix whose columns consist of n independent solutions to the linear system $Ls = 0$, $Ms = 0$, where L and M are a Lax pair of operators and s is a section of the vector bundle E with fibre $\mathbb{C}^n$. Using the same Lax pair of operators as introduced in equation (2.47), we have

$$L = D_w - \zeta D_{\tilde{z}}, \quad M = D_z - \zeta D_{\tilde{w}}. \tag{4.2}$$

so that f on U_+ satisfies $Lf = 0$, $Mf = 0$ for each fixed value of ζ, with its columns forming a basis of sections covariantly constant along the α-planes. The solution $\tilde{f}$ on U_- satisfies the analogous system with ζ replaced by $\tilde{\zeta} = \zeta^{-1}$, as in equations (10.1.1) and (10.1.3) of [1]:

$$\begin{aligned} (D_w - \zeta D_{\tilde{z}})f = 0, \qquad & (D_z - \zeta D_{\tilde{w}})f = 0, \\ (D_{\tilde{z}} - \tilde{\zeta} D_w)\tilde{f} = 0, \qquad & (D_{\tilde{w}} - \tilde{\zeta} D_z)\tilde{f} = 0. \end{aligned} \tag{4.3}$$

The boundary values $h = f|_{\zeta=0}$ and $\tilde{h} = \tilde{f}|_{\zeta=\infty}$ then recover equations (2.48). As observed in [1], a single fundamental solution regular on the entire Riemann sphere would be forced by Liouville's theorem to be independent of ζ, implying a flat connection. The two-patch construction is therefore a necessity for any non-flat ASD connection. Under a gauge transformation $D \mapsto g^{-1}Dg$, the fundamental solutions transform as $f \mapsto g^{-1}f$ and $\tilde{f} \mapsto g^{-1}\tilde{f}$, leaving F unchanged. The patching matrix therefore carries gauge-invariant information about the connection. The map assigning patching data to an ASDYM field is the forward Penrose–Ward transform.

4.2 The Atiyah–Ward Ansatz and Birkhoff Factorisation

The Atiyah–Ward ansatz [23] provides a systematic way to construct solutions to the ASDYM equations from a single scalar function ϕ. Following the treatment in [1], the key observation is that the patching matrix for the holomorphic bundle on $\mathbb{PT}$ can be taken to have the form

$$F = \begin{pmatrix} \zeta & \gamma \\ 0 & \zeta^{-1} \end{pmatrix} \tag{4.4}$$

on the overlap $U_+ \cap U_-$ of two patches of $\mathbb{PT}$, where γ is a holomorphic function on $U_+ \cap U_-$ whose Laurent expansion in the spectral parameter ζ takes the form

$$\gamma = \sum_{i=-\infty}^{\infty} \gamma_{-i}\zeta^i = \gamma_+ + \phi + \gamma_-, \tag{4.5}$$

with γ_+ and γ_- containing only positive and negative powers of ζ respectively, and ϕ the zeroth-order coefficient. The higher coefficients $\gamma_{\pm n}$ are not free, they are determined by the recursion relations

$$\partial_w \gamma_i = \partial_{\tilde{z}} \gamma_{i+1}, \qquad \partial_z \gamma_i = \partial_{\tilde{w}} \gamma_{i+1}. \tag{4.6}$$

These recursion relations arise from the condition that F must be constant along the α-planes, which are those spanned by the vector fields

$$\ell = \partial_w - \zeta \partial_{\tilde{z}}, \qquad m = \partial_z - \zeta \partial_{\tilde{w}}. \tag{4.7}$$

Applying $\ell(\gamma) = 0$ and $m(\gamma) = 0$ and collecting powers of ζ yields the recursion relations (4.6). As noted in [1], it follows from these relations that each coefficient γ_i individually satisfies the scalar wave equation $\Box \gamma_i = 0$, and in particular so does the zeroth-order coefficient ϕ. The Birkhoff factorisation $F = \tilde{f}^{-1} f$ is given by

$$f = \frac{1}{\sqrt{\phi}} \begin{pmatrix} \zeta & \phi + \gamma_+ \\ -1 & -\zeta^{-1}\gamma_+ \end{pmatrix}, \qquad \tilde{f} = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 1 & -\zeta\gamma_- \\ -\zeta^{-1} & \phi + \gamma_- \end{pmatrix}. \tag{4.8}$$

This can be verified directly by computing $\tilde{f}^{-1} f$ and confirming it reproduces F, and by checking that f contains only non-negative powers of ζ and $\tilde{f}$ contains only non-positive powers, establishing holomorphicity on U_+ and U_- respectively. The factorisation Taking the boundary values $h = f|_{\zeta=0}$ and $\tilde{h} = \tilde{f}|_{\zeta=\infty}$, and using the Laurent expansion convention $\gamma = \sum_{i=-\infty}^{\infty} \gamma_{-i} \zeta^i$, we find

$$h = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 0 & \phi \\ -1 & -\gamma_{-1} \end{pmatrix}, \qquad \tilde{h} = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 1 & -\gamma_1 \\ 0 & \phi \end{pmatrix}. \tag{4.9}$$

Their inverses are

$$h^{-1} = \frac{1}{\sqrt{\phi}} \begin{pmatrix} -\gamma_{-1} & -\phi \\ 1 & 0 \end{pmatrix}, \qquad \tilde{h}^{-1} = \frac{1}{\sqrt{\phi}} \begin{pmatrix} \phi & \gamma_1 \\ 0 & 1 \end{pmatrix}. \tag{4.10}$$

It can be shown that Yang's matrix J is then

$$J = \tilde{h}^{-1} h = \frac{1}{\phi} \begin{pmatrix} \phi & \gamma_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \phi \\ -1 & -\gamma_{-1} \end{pmatrix} = \frac{1}{\phi} \begin{pmatrix} -\gamma_1 & \phi^2 - \gamma_1\gamma_{-1} \\ -1 & -\gamma_{-1} \end{pmatrix}. \tag{4.11}$$

The gauge field follows via $\Phi = h\partial h^{-1} + \tilde{h}\tilde{\partial}\tilde{h}^{-1}$. For the first term, we compute

$$h\partial h^{-1} = -\frac{\partial\phi}{2\phi}\mathbf{1} + \frac{1}{\phi} \begin{pmatrix} 0 & 0 \\ \partial\gamma_{-1} & \partial\phi \end{pmatrix} = \frac{1}{\phi} \begin{pmatrix} -\frac{\partial\phi}{2} & 0 \\ \partial\gamma_{-1} & \frac{\partial\phi}{2} \end{pmatrix}, \tag{4.12}$$

and for the second term,

$$\tilde{h}\tilde{\partial}\tilde{h}^{-1} = -\frac{\tilde{\partial}\phi}{2\phi}\mathbf{1} + \frac{1}{\phi} \begin{pmatrix} \tilde{\partial}\phi & \tilde{\partial}\gamma_1 \\ 0 & 0 \end{pmatrix} = \frac{1}{\phi} \begin{pmatrix} \frac{\tilde{\partial}\phi}{2} & \tilde{\partial}\gamma_1 \\ 0 & -\frac{\tilde{\partial}\phi}{2} \end{pmatrix}. \tag{4.13}$$

Adding both terms gives

$$\Phi = h\partial h^{-1} + \tilde{h}\tilde{\partial}\tilde{h}^{-1} = \frac{1}{\phi} \begin{pmatrix} \frac{\tilde{\partial}\phi - \partial\phi}{2} & \tilde{\partial}\gamma_1 \\ \partial\gamma_{-1} & \frac{\partial\phi - \tilde{\partial}\phi}{2} \end{pmatrix}. \tag{4.14}$$

The off-diagonal components are determined by $\gamma_{\pm 1}$ via the recursion relations (4.6), from which one can show that

$$\tilde{\partial}\gamma_1 = 2(\phi_z \mathrm{d}\tilde{w} + \phi_w \mathrm{d}\tilde{z}), \qquad \partial\gamma_{-1} = 2(\phi_{\tilde{z}} \mathrm{d}w + \phi_{\tilde{w}} \mathrm{d}z), \tag{4.15}$$

reproducing the gauge field

$$\Phi_a \mathrm{d}x^a = \frac{1}{2\phi} \begin{pmatrix} \tilde{\partial}\phi - \partial\phi & 2(\phi_z \mathrm{d}\tilde{w} + \phi_w \mathrm{d}\tilde{z}) \\ 2(\phi_{\tilde{z}} \mathrm{d}w + \phi_{\tilde{w}} \mathrm{d}z) & \partial\phi - \tilde{\partial}\phi \end{pmatrix}. \tag{4.16}$$

The ASDYM condition on the resulting gauge field is then equivalent to requiring that ϕ satisfies the wave equation $\Box\phi = 0$.

4.3 Incompatibility of the Atiyah–Ward Ansatz with the Reduction to the Principal Chiral Model

Having established the general form of Yang's matrix for the Atiyah–Ward ansatz, we now ask whether any non-trivial solution constructed in this way is compatible with the symmetry reduction to four-dimensional Chern–Simons theory described in Chapter 3. Recall from equation (3.29) that this reduction requires

$$\mathcal{L}_{\partial_z} J = \mathcal{L}_{\partial_{\bar{z}}} J = 0. \tag{4.17}$$

These conditions are shown to force ϕ to be constant, so that no non-trivial solution constructed via the Atiyah–Ward ansatz admits this reduction. Computing $\mathcal{L}_{\partial_z} J = \partial_z J$ directly from equation (4.11),

$$\partial_z J = -\frac{\partial_z \phi}{\phi^2} \begin{pmatrix} -\gamma_1 & \phi^2 - \gamma_1\gamma_{-1} \\ -1 & -\gamma_{-1} \end{pmatrix} + \frac{1}{\phi} \begin{pmatrix} -\partial_z\gamma_1 & 2\phi\partial_z\phi - \partial_z(\gamma_1\gamma_{-1}) \\ 0 & -\partial_z\gamma_{-1} \end{pmatrix}. \tag{4.18}$$

The $(2,1)$ entry of this expression is

$$(\partial_z J)_{21} = \frac{\partial_z \phi}{\phi^2}. \tag{4.19}$$

For $\mathcal{L}_{\partial_z} J = 0$ we therefore require

$$\partial_z \phi = 0. \tag{4.20}$$

The other components under this condition also force

$$\begin{aligned} \partial_z \gamma_1 &= 0, \\ \partial_z \gamma_{-1} = 0 \implies \partial_{\bar{w}} \phi &= 0, \end{aligned} \tag{4.21}$$

where we have used the recursion relations (4.6). Applying the same argument to $\mathcal{L}_{\partial_{\bar{z}}} J = 0$, the $(2,1)$ entry gives

$$\begin{aligned} \partial_{\bar{z}} \phi &= 0, \\ \partial_{\bar{z}} \gamma_1 = 0 \implies \partial_w \phi &= 0, \\ \partial_{\bar{z}} \gamma_{-1} &= 0. \end{aligned} \tag{4.22}$$

Together these conditions require

$$\partial_z \phi = \partial_{\bar{z}} \phi = \partial_w \phi = \partial_{\bar{w}} \phi = 0, \tag{4.23}$$

which forces ϕ to be a constant. Since $\phi = \text{const}$ corresponds to a flat connection, it follows that no non-trivial solution constructed via the Atiyah–Ward ansatz satisfies the translation invariance condition required for the reduction to a principal chiral model with Wess–Zumino–Witten term as done in [3]. In particular, the BPST instanton, whose scalar function $\phi = 1 + \lambda^2/(x - x_0)^2$ is manifestly non-constant, cannot be reduced to a two-dimensional integrable theory via this chain of correspondences. This also implies that solutions constructed using the Atiyah–Ward ansatz cannot be reduced to a 4d Chern–Simons theory, since the same symmetries are required to do this reduction, just lifted to the twistor space picture. This result motivates studying the instanton solution on its own terms, without attempting the full reduction. In the following section we introduce the BPST instanton explicitly and examine how it sits within the Atiyah–Ward framework.

4.4 The BPST Instanton

The BPST instanton [24] is the charge $k = 1$ SU(2) instanton on Euclidean space $\mathbb{E}^4$. Following the t'Hooft ansatz [25, 26], it is described by the scalar function

$$\phi = 1 + \frac{\lambda^2}{(x - x_0)^2}, \tag{4.24}$$

where $(x - x_0)^2 = |z - z_0|^2 + |w - w_0|^2$ in double null coordinates, λ is the instanton scale parameter, and x_0 is the instanton centre. One can verify directly that ϕ satisfies the four-dimensional Laplace equation

$$\Delta\phi = \partial_z \partial_{\bar{z}} \phi + \partial_w \partial_{\bar{w}} \phi = 0 \tag{4.25}$$

away from $x = x_0$. To see this, we compute

$$\partial_z \partial_{\bar{z}} \left(\frac{\lambda^2}{r^2} \right) = -\frac{\lambda^2}{r^4} + \frac{2\lambda^2 |z - z_0|^2}{r^6}, \qquad \partial_w \partial_{\bar{w}} \left(\frac{\lambda^2}{r^2} \right) = -\frac{\lambda^2}{r^4} + \frac{2\lambda^2 |w - w_0|^2}{r^6}, \tag{4.26}$$

so that

$$\Delta\phi = -\frac{2\lambda^2}{r^4} + \frac{2\lambda^2(|z - z_0|^2 + |w - w_0|^2)}{r^6} = -\frac{2\lambda^2}{r^4} + \frac{2\lambda^2}{r^4} = 0, \tag{4.27}$$

where we used $r^2 = |z - z_0|^2 + |w - w_0|^2$. This confirms that ϕ is a valid input for the Atiyah–Ward ansatz, and the ASDYM equations are satisfied. The gauge field is given by equation (4.16), which under the Euclidean reality conditions $\tilde{z} = \bar{z}$ and $\tilde{w} = -\bar{w}$ gives $\tilde{\partial} = \bar{\partial}$. The derivatives of ϕ are

$$\begin{aligned} \partial_z \phi &= -\frac{\lambda^2(\bar{z} - \bar{z}_0)}{r^4}, \qquad & \partial_{\bar{z}} \phi &= -\frac{\lambda^2(z - z_0)}{r^4}, \\ \partial_w \phi &= -\frac{\lambda^2(\bar{w} - \bar{w}_0)}{r^4}, \qquad & \partial_{\bar{w}} \phi &= -\frac{\lambda^2(w - w_0)}{r^4}. \end{aligned} \tag{4.28}$$

The diagonal components of the gauge field are

$$\pm(\bar{\partial}\phi - \partial\phi) = \pm\frac{\lambda^2}{r^4} \left((z - z_0)\mathrm{d}\bar{z} + (w - w_0)\mathrm{d}\bar{w} - (\bar{z} - \bar{z}_0)\mathrm{d}z - (\bar{w} - \bar{w}_0)\mathrm{d}w \right), \tag{4.29}$$

the top right component is

$$2(\phi_z \mathrm{d}\tilde{w} + \phi_w \mathrm{d}\tilde{z}) = \frac{2\lambda^2}{r^4} \left((\bar{z} - \bar{z}_0)\mathrm{d}\bar{w} - (\bar{w} - \bar{w}_0)\mathrm{d}\bar{z} \right), \tag{4.30}$$

and the bottom left component is

$$2(\phi_{\tilde{z}} \mathrm{d}w + \phi_{\tilde{w}} \mathrm{d}z) = -\frac{2\lambda^2}{r^4} \left((z - z_0)\mathrm{d}w - (w - w_0)\mathrm{d}z \right). \tag{4.31}$$

Setting $z' = z - z_0$, $\bar{z}' = \bar{z} - \bar{z}_0$, $w' = w - w_0$ and $\bar{w}' = \bar{w} - \bar{w}_0$ for brevity, the full gauge field for the BPST instanton is therefore

$$\Phi_a \mathrm{d}x^a = \frac{\lambda^2}{\phi r^4} \begin{pmatrix} z'\mathrm{d}\bar{z} + w'\mathrm{d}\bar{w} - \bar{z}'\mathrm{d}z - \bar{w}'\mathrm{d}w & 2\left(\bar{z}'\mathrm{d}\bar{w} - \bar{w}'\mathrm{d}\bar{z}\right) \\ -2\left(z'\mathrm{d}w - w'\mathrm{d}z\right) & -(z'\mathrm{d}\bar{z} + w'\mathrm{d}\bar{w} - \bar{z}'\mathrm{d}z - \bar{w}'\mathrm{d}w) \end{pmatrix}. \tag{4.32}$$

This is the explicit gauge field for the charge $k = 1$ BPST instanton in double null coordinates, written in terms of the scale parameter λ and instanton centre x_0. It is manifestly traceless and anti-Hermitian as required for an $\mathfrak{su}(2)$-valued connection. The result of Section 4.3 tells us that this solution cannot be reduced to a two-dimensional integrable theory via the Bittleston–Skinner chain of correspondences. This obstruction is not surprising in retrospect. The Atiyah–Ward ansatz parametrises ASDYM solutions by scalar functions satisfying the wave equation, and generic solutions to the wave equation will depend non-trivially on all four coordinates. The translation invariance required by the Bittleston–Skinner reduction is therefore an exceptional condition, not one we should expect a generic solution to satisfy. This suggests that the Atiyah–Ward ansatz, while a powerful tool for constructing ASDYM solutions, is not well-suited to producing solutions compatible with the reduction to integrable two-dimensional models. Finding explicit solutions that do survive the reduction, and understanding what geometric or analytic conditions on the patching matrix F characterise them, remains an interesting open problem. We note that this result appears to be new: while the incompatibility of generic ASDYM solutions with translational symmetry reduction is not surprising, the explicit proof that the recursion relations of the Atiyah–Ward ansatz rule out the entire class does not appear in the literature to the author's knowledge.

5. Conclusion

This dissertation has examined the role of holomorphic Chern–Simons theory on twistor space as a candidate unified origin for integrability, following the foundation built by Costello, Witten, and Yamazaki and further work by Bittleston and Skinner. We have developed the necessary background material, examined the dimensional reduction structure of the theory in detail, and attempted to examine how explicit solutions behave under the reduction chain.

Chapter 2 reviewed all of the relevant background material. We reviewed the notion of integrability and how it is interpreted in different contexts, mostly giving examples of systems that are considered integrable. Anti-self-dual Yang–Mills theory was discussed as a particularly important integrable system, due to its ability to describe other integrable systems through dimensional reductions. We discussed its Lax pair formulation and its equivalent description in terms of Yang's matrix J and the K-matrix, as well as their Lagrangian and Hamiltonian descriptions. Chern–Simons theory was introduced as a topological quantum field theory, and both three-dimensional and four-dimensional theories were studied along with their boundary conditions and edge modes. Finally, the basics of twistor theory were developed, where the twistor correspondence, twistor geometry, and the Penrose transform were discussed in detail.

Chapter 3 presented the central construction of the dissertation, which started with the formulation of holomorphic Chern–Simons theory on twistor space. We then showed the dimensional reduction of this to an effective description of the four dimensional Wess–Zumino–Witten model, following the work of Bittleston and Skinner in [3]. The equation of motion of the WZW_4 model is Yang's equation, which is also the on-shell condition of the anti-self-dual Yang–Mills equation; the two theories are therefore classically equivalent on-shell, sharing the same space of solutions parametrised by Yang's matrix. We then followed their further reduction of this to the two dimensional principal chiral model. Implementing the same symmetry reduction directly at the level of holomorphic Chern–Simons theory on twistor space, the spectral parameter emerging geometrically from the $\mathbb{CP}^1$ fibration, we recovered four-dimensional Chern–Simons theory with a meromorphic one-form ω whose poles and zeros are inherited from the twistor geometry. This could then also be further reduced to the principal chiral model, showing that doing the same reductions in opposite orders reproduced the same result. The symmetry reduction from holomorphic Chern–Simons to four-dimensional Chern–Simons theory was verified explicitly at the level of the Lagrangian, filling in details of the computation in [3].

Chapter 4 then turned to tracing explicit solutions through this reduction chain. The Penrose–Ward transform was reviewed through the language of patching matrices, before explicitly looking at using the Atiyah–Ward ansatz to construct solutions to be brought through the chain. The main result of this chapter is original: we prove that the Atiyah–Ward ansatz is incompatible with the Bittleston–Skinner reduction chain. Specifically, the condition of translational invariance in two directions forced all the scalar potential ϕ to be constant, meaning that non-trivial solutions could not be reduced in the way shown in [3]. As a consequence, solutions constructed using the Atiyah–Ward ansatz cannot be reduced to a two-dimensional integrable theory via the Bittleston–Skinner chain of correspondences. The BPST instanton gauge field was computed explicitly in double null coordinates as an illustration of the ansatz.

The no-go result of Chapter 4, while not claiming to be a deep theorem, is to the author's knowledge new. It closes off a natural first avenue for constructing explicit solutions compatible with the Bittleston–Skinner correspondence, and identifies precisely where the obstruction lies: the re-

cursion relations of the Atiyah–Ward ansatz are incompatible with translation invariance in two directions for any non-constant scalar potential. The result of leads one to wonder what types of solutions can be carried through the chain of correspondences established by Bittleston and Skinner. The Atiyah–Ward ansatz parametrises ASDYM solutions by scalar functions depending generically on all four space-time coordinates, whereas the Bittleston–Skinner reduction requires translation invariance in two of them. The two conditions are simply incompatible for non-trivial solutions. Solutions compatible with the reduction must be sought through patching matrices of a form different from the Atiyah–Ward class. Future work would likely involve looking specifically at solutions that have translational invariance in two directions already present within them, as well as understanding what conditions on the patching matrix F characterise such solutions.

More broadly, the work done on this dissertation illustrates the power and potential of the twistor theory approach to integrability. Holomorphic Chern–Simons theory provides an action principle on twistor space that has already been shown to act as a unified origin for lower dimensional integrable models. Through considering different dimensional reductions, perhaps by symmetries such as rotations or a three-dimensional translational reduction, one may be able to find different known integrable models or even novel ones. At the same time, carrying explicit solutions through the dimensional reduction chain is non-trivial, and the obstruction identified in Chapter 4 highlights that not every solution of the higher-dimensional theory descends to a solution of the lower-dimensional one. Understanding the precise image of the reduction map, and constructing explicit solutions to integrable models directly from twistor data, remain important open directions in this programme. We do not wish to claim that holomorphic Chern-Simons theory provides a universal origin for all integrable models, but nonetheless twistor theory's role in integrability will surely generate much discussion, interest and excitement in the near future.

Appendix A

Proof of quantized nature of WZW term in 3dCS action variation

Let M^3 be a closed, oriented three–manifold and let

$$g : M^3 \to G \tag{A.1}$$

be a smooth map into a compact simple Lie group G (e.g. $SU(N)$). Define the Lie algebra–valued one–form

$$\beta := g^{-1}\mathrm{d}g. \tag{A.2}$$

We wish to show that

$$W[g] \;:=\; \frac{1}{24\pi^2}\int_{M^3}\mathrm{tr}(\beta\wedge\beta\wedge\beta) \in \mathbb{Z} \tag{A.3}$$

Let $g_t : M^3 \to G$ be a smooth one-parameter family of maps, $t \in [0,1]$, and define $\beta_t := g_t^{-1}dg_t$. Then it is our aim to prove that

$$W[g_t] := \frac{1}{24\pi^2}\int_{M^3}\mathrm{tr}(\beta_t^3) \tag{A.4}$$

is independent of t. In particular, we want to prove that $W[g]$ depends only on the homotopy class of g. We start by differentiating under the integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}W[g_t] = \frac{1}{24\pi^2}\int_{M^3}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{tr}(\beta_t^3) = \frac{3}{24\pi^2}\int_{M^3}\mathrm{tr}\big((\partial_t\beta_t)\wedge\beta_t\wedge\beta_t\big). \tag{A.5}$$

Set $X_t := g_t^{-1}\partial_t g_t$ (a Lie-algebra–valued 0–form). A short computation using gives

$$\partial_t\beta_t = \mathrm{d}X_t + [\beta_t, X_t] =: D_tX_t, \tag{A.6}$$

the covariant derivative of X_t with respect to the connection β_t. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}W[g_t] = \frac{3}{24\pi^2}\int_{M^3}\mathrm{tr}\big((\mathrm{d}X_t + [\beta_t, X_t])\wedge\beta_t\wedge\beta_t\big). \tag{A.7}$$

The commutator term does not contribute because the trace is cyclic:

$$\mathrm{tr}([\beta_t, X_t]\wedge\beta_t\wedge\beta_t) = 0. \tag{A.8}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}W[g_t] = \frac{3}{24\pi^2}\int_{M^3}\mathrm{tr}(\mathrm{d}X_t\wedge\beta_t\wedge\beta_t). \tag{A.9}$$

Now use the Leibniz rule and the flatness of the Maurer–Cartan form ($\mathrm{d}\beta_t + \beta_t\wedge\beta_t = 0$) to write

$$\mathrm{d}\,\mathrm{tr}(X_t\beta_t\wedge\beta_t) = \mathrm{tr}(\mathrm{d}X_t\wedge\beta_t\wedge\beta_t) + \mathrm{tr}(X_t\,\mathrm{d}(\beta_t\wedge\beta_t)), \tag{A.10}$$

but $\mathrm{d}(\beta_t \wedge \beta_t)$ is proportional to the curvature and vanishes for $\beta_t = g_t^{-1}\mathrm{d}g_t$, so

$$\mathrm{d}\ \mathrm{tr}(X_t \beta_t \wedge \beta_t) = \mathrm{tr}(\mathrm{d}X_t \wedge \beta_t \wedge \beta_t). \tag{A.11}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t} W[g_t] = \frac{3}{24\pi^2} \int_{M^3} \mathrm{d}\ \mathrm{tr}(X_t \beta_t \wedge \beta_t). \tag{A.12}$$

Since M^3 is closed, the integral of an exact form vanishes by Stokes' theorem, so

$$\frac{\mathrm{d}}{\mathrm{d}t} W[g_t] = 0. \tag{A.13}$$

Thus $W[g_t]$ is constant along the homotopy and therefore depends only on the homotopy class of g. For any compact simple Lie group G, the third homotopy group is

$$\pi_3(G) \cong \mathbb{Z}. \tag{A.14}$$

Hence any map $g : S^3 \to G$ falls into a homotopy class labelled by an integer n, the *degree* or *winding number*. By homotopy invariance, $W[g]$ can only depend on this integer. We can show this explicitly for $G = SU(2)$ and $M_3 = S^3$ [27]. Under the identity transformation $g = e$, $W[e] = 0$ trivially. Now consider the 1-form on S^3 defined by

$$\beta_1 = \frac{1}{r^2}\,(x^4 - i x^k \sigma_k)\,\mathrm{d}(x^4 + i x^\ell \sigma_\ell), \tag{A.15}$$

where (x^1, x^2, x^3, x^4) are coordinates on the unit $S^3 \subset \mathbb{R}^4$, $r^2 = (x^4)^2 + x_k x^k$, and σ_k are the Pauli matrices. Since the map $g_1 : S^3 \to SU(2) \cong S^3$ covers the group manifold uniformly, the integrand $\mathrm{tr}(\beta_1 \wedge \beta_1 \wedge \beta_1)$ is constant on S^3. Thus we may evaluate it at the north pole

$$(x^4, x^1, x^2, x^3) = (1, 0, 0, 0). \tag{A.16}$$

At this point one finds

$$\beta_1 = i\,\sigma_k\,\mathrm{d}x^k. \tag{A.17}$$

Therefore

$$\begin{aligned} \mathrm{tr}(\beta_1 \wedge \beta_1 \wedge \beta_1) &= i^3\ \mathrm{tr}(\sigma_i \sigma_j \sigma_k)\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\ &= 2\,\epsilon_{ijk}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\ &= 12\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \end{aligned} \tag{A.18}$$

The coordinates (x^1, x^2, x^3) serve as valid local coordinates on each hemisphere of S^3, and the 3–form

$$\omega := \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{A.19}$$

gives the volume form at the north pole. Using (A.18),

$$\int_{S^3} \mathrm{tr}(\beta_1 \wedge \beta_1 \wedge \beta_1) = 12 \int_{S^3} \omega = 12\,(2\pi^2) = 24\pi^2, \tag{A.20}$$

where $2\pi^2$ is the area of the unit S^3. Hence

$$\frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}(\beta_1 \wedge \beta_1 \wedge \beta_1) \ = \ 1. \tag{A.21}$$

Now consider the map $g_n : S^3 \to SU(2)$ defined by the n–fold composition

$$g_n = (g_1)^n. \tag{A.22}$$

As in the standard argument, decompose S^3 into northern and southern hemispheres $U_N^{(3)}$ and $U_S^{(3)}$. A map g_1 of winding number 1 can always be smoothly deformed into maps g_1^N and g_1^S with the same winding number such that

$$g_1^N(x) = e \quad \text{for } x \in U_S^{(3)}, \qquad g_1^S(x) = e \quad \text{for } x \in U_N^{(3)}. \tag{A.23}$$

Under this deformation, the composite map $g_2 = g_1 g_1$ becomes

$$g_2(x) = \begin{cases} g_1^N(x), & x \in U_N^{(3)}, \\ g_1^S(x), & x \in U_S^{(3)}. \end{cases} \tag{A.24}$$

Let

$$\beta_2 := g_2^{-1} \mathrm{d} g_2. \tag{A.25}$$

Then

$$\begin{aligned} \frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}(\beta_2{}^3) &= \frac{1}{24\pi^2} \left(\int_{U_N^{(3)}} \mathrm{tr}\left([(g_1^N)^{-1} \mathrm{d} g_1^N]^3 \right) + \int_{U_S^{(3)}} \mathrm{tr}\left([(g_1^S)^{-1} \mathrm{d} g_1^S]^3 \right) \right) \\ &= 1 + 1 = 2. \end{aligned} \tag{A.26}$$

Repeating this argument for $g_n = (g_1)^n$ gives

$$\frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}\big[(g_n^{-1} \mathrm{d} g_n)^3 \big] = n. \tag{A.27}$$

This is precisely the winding number for $g : S^3 \to SU(2)$. This is an explicit example of the fact that for a compact simple Lie group G, one has $\pi_3(G) \cong \mathbb{Z}$, and $W[g]$ computes this integer for $g : S^3 \to G$. For a general closed, oriented three–manifold M^3, the closed three–form

$$\omega_3 := \frac{1}{24\pi^2} \mathrm{tr}(\beta \wedge \beta \wedge \beta), \qquad \beta = g^{-1} \mathrm{d} g, \tag{A.28}$$

defines an integral cohomology class in $H^3(G, \mathbb{Z})$. For any smooth map $g : M^3 \to G$, the pullback $g^* \omega_3$ is a closed three–form on M^3, and the pairing

$$\int_{M^3} g^* \omega_3 \tag{A.29}$$

with the fundamental class $[M^3]$ is therefore an integer. In other words,

$$\frac{1}{24\pi^2} \int_{M^3} \mathrm{tr}(g^{-1} \mathrm{d} g \wedge g^{-1} \mathrm{d} g \wedge g^{-1} \mathrm{d} g) \in \mathbb{Z} \tag{A.30}$$

for any closed, oriented M^3 and any smooth $g : M^3 \to G$. In particular,

$$\frac{1}{3} \int_{M^3} \mathrm{tr}(g^{-1} \mathrm{d} g)^3 \;=\; 8\pi^2 \, W[g] \;\in\; 8\pi^2 \mathbb{Z}. \tag{A.31}$$

□

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