Keldysh Field Theory for Lattice Population Dynamics

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Hamilton Trust Project

July 2, 2026

Keldysh Field Theory of Non-Equilibrium Systems

The Contour Idea

Expectation Value in the Schrödinger Picture

$$O(t) = \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \langle \phi(t)| \hat{O}(t) |\phi(t)\rangle \tag{1}$$

$$= \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \langle \phi(t_0)| \hat{U}(t_0, t) \hat{O}(t) \hat{U}(t, t_0) |\phi(t_0)\rangle \,. \tag{2}$$

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$$\int d[\bar{\phi}, \phi] \equiv \int \frac{1}{\pi} d\,\mathrm{Re}\{\phi\}\,\mathrm{Im}\{\phi\} \tag{3}$$

The Contour Idea

Expectation Value in the Schrödinger Picture

$$\langle\phi(t_0)|\,\tilde{T}\left[\exp\left\{\int_t^{t_0} d\bar{t}\hat{H}(\bar{t})\right\}\right]\hat{O}(t)T\left[\exp\left\{\int_{t_0}^{t} d\bar{t}\hat{H}(\bar{t})\right\}\right]|\phi(t_0)\rangle\,. \qquad (1)$$

The Contour Idea

Expectation Value in the Schrödinger Picture

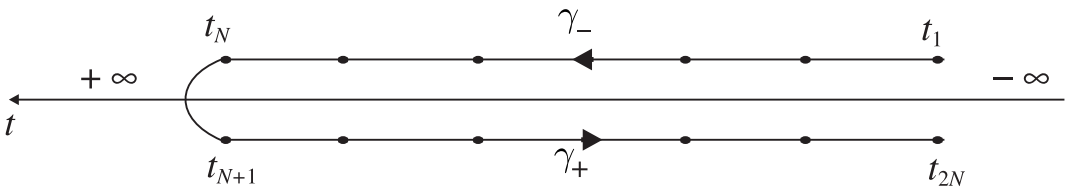
$$\tilde{T}\left[\hat{H}(t_1)\dots\hat{H}(t_n)\right]\hat{O}(t)T\left[\hat{H}(t'_1)\dots\hat{H}(t'_m)\right]. \tag{1}$$

Keldysh Contour

$$\gamma_K = \gamma_+ \oplus \gamma_-$$

Keldysh Contour

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Keldysh Contour

Expectation Value on the Contour

$$T_{\mathcal{C}} \left[\hat{H}(t_{1-}) \dots \hat{H}(t_{n-}) \hat{O}(t) \hat{H}(t'_{1+}) \dots \hat{H}(t'_{m+}) \right]. \tag{2}$$

Keldysh Contour

Contour Evolution Operator

$$\hat{\mathcal{U}}_{\mathcal{C}}(t,t_0) = T_{\mathcal{C}}\left[\exp\left\{\int_{t_0}^{t} d\bar{t}\hat{H}(\bar{t})\right\}\right]. \tag{2}$$

Non-Equilibrium Partition Function

Initial Density Matrix

$$\hat{\rho} = \hat{\rho}(\hat{H}). \tag{3}$$

Non-Equilibrium Partition Function

Canonical Ensemble of States

$$\hat{\rho}_0 = \frac{e^{\beta \hat{H}}}{Z}. \tag{3}$$

Non-Equilibrium Partition Function

Canonical Partition Function

$$Z = \mathrm{Tr}\left\{e^{\beta \hat{H}}\right\}. \tag{3}$$

Non-Equilibrium Partition Function

Non-Equilibrium Partition Function

$$Z = \frac{\mathrm{Tr}\left\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}_0\right\}}{\mathrm{Tr}\{\hat{\rho}_0\}}. \tag{3}$$

Green's Functions

Partition Function

$$Z = \frac{1}{\mathrm{Tr}\{\rho_0\}} \int \prod_j^{2N} d[\bar{\phi}_j, \phi_j] \exp\left\{ i \sum_{j,j'} \bar{\phi}_j G^{-1}_{jj'} \phi_{j'} \right\}. \tag{4}$$

Green's Functions

The Continuum Limit

In the continuum limit we take $N \to \infty$ and $\phi_j \to \phi(t)$

$$Z = \int \mathbf{D}[\bar{\phi}(t), \phi(t)] e^{iS[\bar{\phi},\phi]} \tag{4}$$

where $\mathbf{D}[\bar{\phi}(t), \phi(t)]$ is shorthand for $\prod_j^{2N} d[\bar{\phi}_j, \phi_j]$.

The Action

The Action

$$S = \int_{\mathcal{C}} dt \; \vec{\phi}^{\dagger} G^{-1} \vec{\phi}, \tag{5}$$

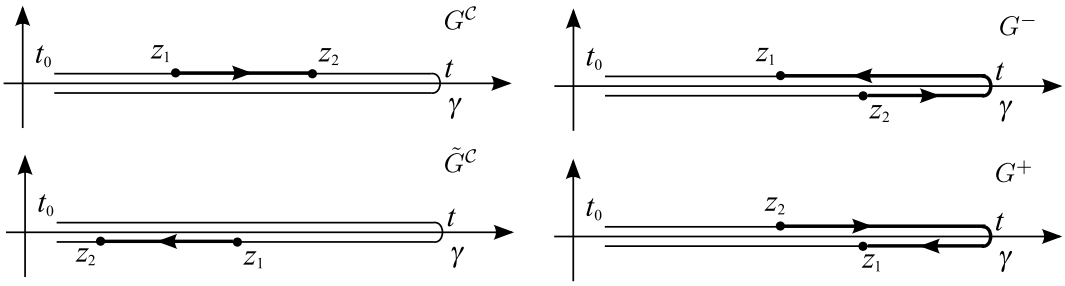
where $\vec{\phi}^{\dagger} = (\bar{\phi}_+, \bar{\phi}_-)$. $\pm$ for ϕ on $\gamma_{\pm}$.

Green's Function Matrix

Green's Functions on Forward and Backward Contours

$$G = \begin{pmatrix} G^{\mathcal{C}} & G^{-} \\ \tilde{G}^{\mathcal{C}} & G^{+} \end{pmatrix} \tag{6}$$

Green's Function Diagrams



The type of points that are related by each Green's function. The points z_i are temporal points on the complex plane.

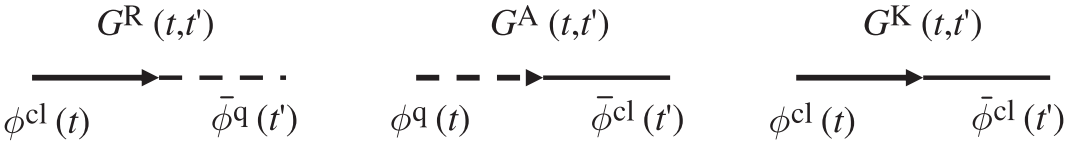
Keldysh Rotation

Keldysh Rotation Matrix

By the transformation

$$\frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) G \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix}. \tag{7}$$

Keldysh Rotation



Keldysh Rotation

Keldysh Symmetry

$$S[\phi^{\mathsf{cl}}, 0] = 0 \tag{7}$$

Reflects fact that a purely classical system will experience no fluctuations, but this can be used to model classical stochastic noise!

Keldysh Rotation

Keldysh Action

$$S = \int_{\mathcal{C}} dt \; \vec{\phi}_K^{\dagger} G^{-1} \vec{\phi}_K. \tag{7}$$

where

$$\vec{\phi}_K = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_+ + \phi_- \\ \phi_+ - \phi_- \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \phi^{\mathsf{cl}} \\ \phi^{\mathsf{q}} \end{pmatrix}. \tag{8}$$

Connection to Classical Stochastic Systems

Real Keldysh Rotation

$$\phi(t) = aX(t) + ibP(t). \tag{9}$$

We can repeat the procedure above with real field $X(t)$

Connection to Classical Stochastic Systems

Feynman-Keldysh Action

$$S = \int dt\ \vec{X}^T D^{-1} \vec{X} \tag{9}$$

where

$$\vec{X} = \frac{1}{4}\begin{pmatrix} X^+ + X^- \\ X^+ - X^- \end{pmatrix} \equiv \begin{pmatrix} X^{\mathsf{cl}} \\ X^{\mathsf{q}} \end{pmatrix} \qquad D = \begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix} \tag{10}$$

From Keldysh to Classical Stochastic Dynamics

The Classical Limit

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The Langevin Equation

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The Fokker–Planck Equation

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Population Dynamics

The Classical Limit

↓

The Langevin Equation

↓

The Fokker–Planck Equation

↓

Population Dynamics

Classical Limit of the Keldysh Action

- Assuming that the **quantum fluctuations** are small ($X^{\mathsf{q}} \ll X^{\mathsf{cl}}$) the **Feynman-Keldysh action** [1] reduces to a classical dissipative action of a particle connected to an **Ohmic bath**,

$$S_{\mathsf{diss}}[X^{\mathsf{cl}}, X^{\mathsf{q}}] = \int_{-\infty}^{+\infty} \mathrm{d}t \, \left[-2X^{\mathsf{q}} \left(\ddot{X}^{\mathsf{cl}} + \gamma \dot{X}^{\mathsf{cl}} + V'(X^{\mathsf{cl}}) \right) + 4i\gamma T (X^{\mathsf{q}})^2 \right].$$

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- First term: deterministic dynamics.
- Second term: thermal fluctuations.
- $S[X^{\mathsf{cl}}, 0] = 0$ (Keldysh symmetry) ensures probability conservation.
- Classical limit: quantum fluctuations are neglected while thermal fluctuations remain.

The Classical Limit

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The Langevin Equation

↓

The Fokker–Planck Equation

↓

Population Dynamics

The Langevin Equation

- Applying a Hubbard-Stratonovich transformation [1] introduces an auxiliary Gaussian noise field, $\xi(t)$, yielding the Langevin equation:

$$\ddot{X}^{\mathsf{cl}} = -\gamma \dot{X}^{\mathsf{cl}} - V'(X^{\mathsf{cl}}) + \xi(t).$$

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- Newton's equation acquires:
  - A friction force $(-\gamma \dot{X})$.
  - A stochastic force $(\xi(t))$.
- Gaussian white noise satisfies

$$\langle \xi(t)\xi(t') \rangle = 2\gamma T \delta(t - t').$$

The Classical Limit

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The Fokker–Planck Equation

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Population Dynamics

The Fokker–Planck Equation

- The Fokker–Planck equation evolves the probability distribution rather than individual trajectories,

$$\partial_t \mathcal{P}(X,t) = -\partial_X \left[A(X)\mathcal{P}(X,t) - \partial_X [D(X)\mathcal{P}(X,t)]\right].$$

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$$\partial_t \Psi(x,t) = -T[-\partial_x^2 + W(x)]\Psi(x,t).$$

- The stationary distribution here corresponds to the ground state of $\hat{H}_{\mathsf{Sch}}$.

The Classical Limit

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The Langevin Equation

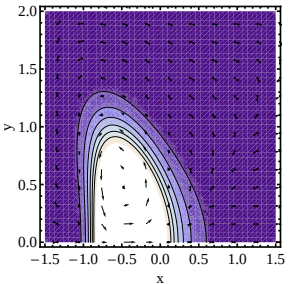
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Population Dynamics

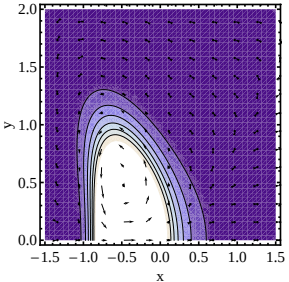
Noise-Induced Stabilisation



Quasi-stationary probability distribution from [2], with x the total population deviation and y the sub-population imbalance.

- Probability contours together with the probability current.

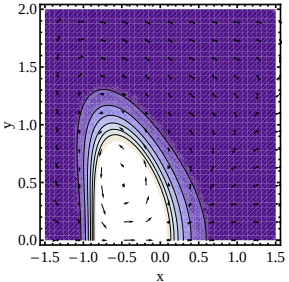
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Noise-Induced Stabilisation



Quasi-stationary probability distribution from [2], with x the total population deviation and y the sub-population imbalance.

- Probability contours together with the probability current.
- The arrows represent the probability current density.
- Noise can stabilise states that would not exist in the deterministic dynamics.

The Bose–Hubbard Model [3]

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1) - \mu \sum_i \hat{n}_i - \sum_i \epsilon_i \hat{n}_i \tag{11}$$

- J: hopping amplitude between neighbouring sites, models migration.

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- U: on-site interaction energy, models local competition ($U > 0$) or cooperation ($U < 0$).
- μ: chemical potential, controls mean population.
- ϵ_i: site-dependent energy offset, models environmental heterogeneity across the lattice.

Future Work: Towards Population Dynamics

Bose–Hubbard Model

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Lattice Hamiltonian

↓

Obtain Green's Functions

↓

Fit Parameters

↓

Predict Population Dynamics

Conclusions

- We have developed the theoretical framework connecting Keldysh field theory to classical stochastic dynamics.

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- We have developed the theoretical framework connecting Keldysh field theory to classical stochastic dynamics.
- The Langevin equation describes individual stochastic trajectories.
- The Fokker–Planck equation describes the evolution of the full probability distribution.
- These techniques provide a framework for modelling stochastic population dynamics on lattices.
- The next step is computing Green's functions for the Bose-Hubbard model and fitting parameters to real population data.

Questions?

References

1. A. Kamenev, *Field Theory of Non-Equilibrium Systems*, (Cambridge University Press, Cambridge, 2011), ISBN: 978-0-521-76082-9.
2. M. Parker *et al.*, *Physical Review Letters* **107**, 180603 (2011).
3. M. P. A. Fisher *et al.*, *Physical Review B* **40**, 546–570 (1989).

Appendix

Bosonic Coherent States

Many-Body Bosonic Hilbert Space

$$\hat{b}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad \hat{b}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{12}$$

Bosonic Coherent States

Coherent States

$$\hat{b}\,|\phi\rangle = \phi\,|\phi\rangle\,, \qquad \langle\phi|\,\hat{b}^\dagger\bar{\phi}.$$
$$|\phi\rangle = \sum_n \frac{\phi^n}{\sqrt{n!}}\,|n\rangle = \sum_n \frac{\phi^n}{\sqrt{n!}}\frac{(\hat{b}^\dagger)^n}{\sqrt{n!}}\,|0\rangle = e^{\phi\hat{b}^\dagger}\,|0\rangle\,. \tag{12}$$

Bosonic Coherent States

Unity

$$\hat{1} = \int d[\bar{\phi}, \phi] e^{-|\phi|^2} |\phi\rangle \langle\phi| . \tag{12}$$

$d[\bar{\phi}, \phi] = d\,\mathrm{Re}\{\phi\} d\,\mathrm{Im}\{\phi\}/\pi$

Green's Functions

Discretely Evolving the System

We break the contour into $2N-2$ intervals of length δ_t. Insert unity for field at each time step t_j

$$\hat{1} = \int d[\bar{\phi}_j, \phi_j] e^{-|\phi|^2} |\phi_j\rangle \langle\phi_j| . \tag{13}$$

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For $N = 2$ we have

$$\mathrm{Tr}\{\hat{\mathcal{U}}_C \rho_0\} = \langle\phi_4| \hat{\mathcal{U}}_{\delta_{t_-}} |\phi_3\rangle \langle\phi_3| \hat{1} |\phi_1\rangle \langle\phi_2| \hat{\mathcal{U}}_{\delta_{t_+}} |\phi_1\rangle \langle\phi_1| \hat{\rho}_0 |\phi_4\rangle \tag{14}$$

Green's Functions

Small Time Step Evolution Operator

One can show using $\langle\phi|\,\hat{H}(\hat{b}^\dagger,\hat{b})\,|\phi'\rangle = e^{\phi\phi'}H(\bar{\phi},\phi)$

$$\langle\phi|\,\hat{\mathcal{U}}_{\delta_{t_\pm}}\,|\phi'\rangle = \langle\phi|\,e^{\mp i\hat{H}(\hat{b}^\dagger,\hat{b})}\,|\phi'\rangle \approx \langle\phi|\,\hat{1}\mp\hat{H}(\hat{b}^\dagger,\hat{b})\,|\phi'\rangle \tag{13}$$