Hamilton Trust Report

Keldysh Field Theory for Lattice Population Dynamics

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1 Introduction

The study of non-equilibrium systems and stochastic methods has long lent itself to the study of dynamics of natural systems. Particularly in the case of *metastable states*, which can describe ecological systems [1], plant biology [2] as well as model competing gene alleles [3].

The goal set forth by this project was to investigate the application of stochastic and field-theoretic methods, in particular using the *Keldysh formalism*, to model population density dynamics on a lattice. In this report we review how the Keldysh formalism can be used to obtain Green's functions in frequency-momentum space. Taking the classical limit, we can obtain a *Langevin equation* for the system, which is subject to a random force i.e white noise. We then show how by searching for the *"Schrödinger equation"* of a system described by the Langevin equations gives rise to the *Fokker-Planck equations*.

Finally, we review how noise stabilization arises from the Langevin equations by looking at some example models. We go on to discuss how model for human immigration, emigration, birth, and death could be constructed in a similar way. A bulk of this report details how Green's functions are obtained in the Keldysh formalism. Green's functions are a natural tool in dynamical systems as they encode the relationships between points in different points in space and time. In this way we hope to use a model quantum Hamiltonian, namely the *Bose-Hubbard Hamiltonian* [4], to model people and their movements between sites on a lattice. The choice of the Bose-Hubbard Hamiltonian is motivated by the encoding of per-lattice site dynamics, Coulombic repulsion from lattice sites proportional to particle number, and bosonic interactions as this will not limit the number of people per lattice site.

2 Green's Functions

Green's functions are key objects of study in condensed matter theory. They contain the information on the connection between two points in spacetime. In this way they are crucial to understand in the study of non-equilibrium systems. First, we will define our Green's functions in the *Keldysh path integral formalism*. This will allow us to easily define the relevant action that ties the formalism to classical stochastic equations. To do this we will briefly revise some key concepts such as, coherent states, and partition functions. We will detail this formalism exclusively for bosons because, as discussed, these will be the ideal candidates for the modelling of human populations later.

2.1 Bosonic Coherent States

Consider a single quantum level in a many-body bosonic state, occupied by n particles $|n\rangle$. These states form a orthonormal basis $\langle n|n'\rangle = \delta_{nn'}$, and have the usual completeness relation $\sum_n |n\rangle\langle n| = \hat{1}$. We let $\hat{b}, \hat{b}^\dagger$ be the annihilation and creation operators that act on this many-body Hilbert space, subject to the usual rules

$$\hat{b}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad \hat{b}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{2.1.1}$$

Coherent states are defined as the eigenstates of the annihilation operator

$$\hat{b}\,|\phi\rangle = \phi\,|\phi\rangle\,, \qquad \langle\phi|\,\hat{b}^\dagger = \langle\phi|\,\bar{\phi}. \tag{2.1.2}$$

For any *normally ordered Hamiltonian* we have

$$\langle\phi|\,\hat{H}(\hat{b}^\dagger,\hat{b})\,|\phi'\rangle = H(\bar{\phi},\phi)\,\langle\phi|\phi'\rangle\,. \tag{2.1.3}$$

One may check

$$|\phi\rangle = \sum_n \frac{\phi^n}{\sqrt{n!}}\,|n\rangle = e^{\phi\hat{b}^\dagger}\,|0\rangle\,, \tag{2.1.4}$$

which implies that the overlap of two coherent states is

$$\langle\phi|\phi'\rangle = e^{\bar{\phi}\phi'}. \tag{2.1.5}$$

Using the following identity of the *Gaussian Integral* [5]

$$I[\bar{J},J] = \int d[\bar{\phi},\phi]e^{-\bar{\phi}\phi+\bar{\phi}J+\bar{J}\phi} = e^{\bar{J}J}, \tag{2.1.6}$$

with $d[\bar{\phi},\phi] \equiv d\,\mathrm{Im}\{\phi\}d\,\mathrm{Re}\{\phi\}/\pi$, we can re-express unity in terms of the coherent basis

$$\begin{aligned} &\int d[\bar{\phi},\phi]e^{-|\phi|^2}\bar{\phi}^n\phi^{n'} = \frac{\partial^{n+n'}}{\partial J^n\partial\bar{J}^{n'}}I[\bar{J},J]\bigg|_{\bar{J}=J=0} = n!\delta_{n,n'},\\ \Rightarrow &\int d[\bar{\phi},\phi]e^{-|\phi|^2}\,|\phi\rangle\,\langle\phi| \overset{\text{by equation (2.1.4)}}{=} \hat{1}. \end{aligned} \tag{2.1.7}$$

Using the above we can define the trace of an operator in the usual way

$$\mathrm{Tr}\big\{\hat{O}\big\} \equiv \sum_n \langle n|\,\hat{O}\,|n\rangle = \int d[\bar{\phi},\phi]e^{-|\phi|^2}\,\langle\phi|\,\hat{O}\,|\phi\rangle\,, \tag{2.1.8}$$

where the operator $\hat{O}$ is in the *Schrödinger picture*.

2.2 The Contour Idea

The *Keldysh contour* is the central idea of the Keldysh formalism. We motivate its introduction by considering a many-body system that obeys a time-dependent Hamiltonian $\hat{H}(t)$. For the moment we assume that we know the system at t_0 was initially in a state defined by the many-body density matrix $\hat{\rho}_0$ as well as assuming that there were no interactions between the particles at time t_0. The interactions are then adiabatically switched on some time before our observation. In this way the density matrix is driven away from equilibrium. The dynamics of the density matrix are governed by the *Von Neumann equation*

$$\partial_t \hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)], \tag{2.2.1}$$

with $\hbar = 1$. This is formally solved with the help of a unitary evolution operator $\hat{\mathcal{U}}(t, t_0)$, which takes the system from time t_0 to time t. By using the fact that the unitary operator obeys the *Schrödinger equation* we find

$$\hat{\mathcal{U}}(t, t_0) = T\left[\exp\left\{-i\int_{t_0}^{t} d\bar{t}\, \hat{H}(\bar{t})\right\}\right], \tag{2.2.2}$$

where T is the time ordering operator. For the time-dependent expectation value of an operator $\hat{O}$ we can see

$$\begin{aligned} O(t) \equiv \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \, \langle \phi(t)| \, \hat{O}(t) \, |\phi(t)\rangle &= \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \, \langle \phi(t_0)| \, \hat{\mathcal{U}}(t_0, t)\hat{O}(t)\hat{\mathcal{U}}(t, t_0) \, |\phi(t_0)\rangle \\ = \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \, \langle \phi(t_0)| \, \tilde{T}\left[\exp\left\{\int_{t}^{t_0} d\bar{t}\hat{H}(\bar{t})\right\}\right] \hat{O}(t) T\left[\exp\left\{\int_{t_0}^{t} d\bar{t}\hat{H}(\bar{t})\right\}\right] |\phi(t_0)\rangle \,, \end{aligned} \tag{2.2.3}$$

where $\tilde{T}$ is the reverse time-ordering operator. By expanding the exponentials in equation (2.2.3) we see

$$\tilde{T}\left[\hat{H}(t_1)\ldots\hat{H}(t_n)\right]\hat{O}(t)T\left[\hat{H}(t'_1)\ldots\hat{H}(t'_m)\right]. \tag{2.2.4}$$

We can more conveniently write the above by introducing the notion of the *Keldysh-Schwinger contour*

$$\gamma_K = \gamma_+ \oplus \gamma_-, \tag{2.2.5}$$

where γ_- runs from (t_0, t), and γ_+ runs from (t, t_0) figure 2.2.1. This allows us to define operators with respect to the contour $\mathcal{C}$. In this way we rewrite equation (2.2.4)

$$T_{\mathcal{C}}\left[\hat{H}(t_1)\ldots\hat{H}(t_n)\hat{O}(t)\hat{H}(t'_1)\ldots\hat{H}(t'_m)\right], \tag{2.2.6}$$

where the operator $T_{\mathcal{C}}$, is the time ordering operator on the contour. We also can then define the contour evolution operator

$$\hat{\mathcal{U}}_{\mathcal{C}}(z, z_0) = T_{\mathcal{C}}\left[\exp\left\{-i\int_{z_0}^{z} d\bar{t}\ \hat{H}(\bar{t})\right\}\right], \tag{2.2.7}$$

where z_0 and z are used to highlight that the times are now complex. In context, z and t will be used interchangeably, to mean time on the contour.

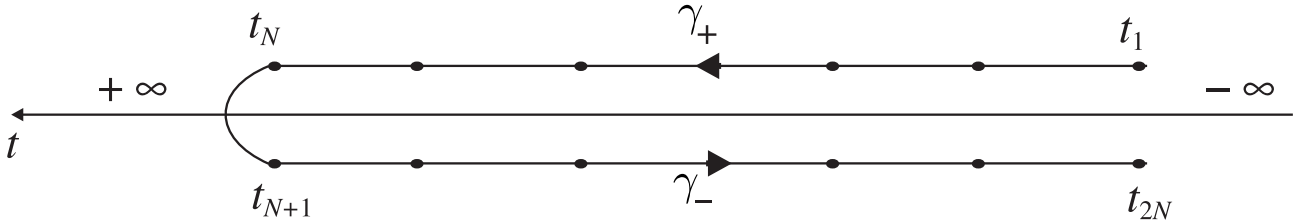


Figure 2.2.1: The Keldysh-Schwinger contour is an oriented contour in the complex time domain.

2.3 Non-Equilibrium Partition Function

The initial density matrix can be chosen depending on what information is available to you about a given system. For a generic system where the initial state is not known, like in experiment, one can pick the initial density matrix to be taken over an ensemble of states

$$\hat{\rho}_0 = \frac{e^{-\beta \hat{H}}}{Z} \tag{2.3.1}$$

where Z is the partition function. Over the entire system the partition function takes the form

$$\boxed{Z = \frac{\mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}_0\}}{\mathrm{Tr}\{\hat{\rho}_0\}}.} \tag{2.3.2}$$

The structure of equation (2.3.2) can be discerned by splitting the contour γ_K into $2N-2$ intervals of length δ_t, (N time points per branch). By inserting unity in-between each time step we see, for $N = 2$

$$\mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}_0\} = \langle\phi_4|\,\hat{\mathcal{U}}_{\delta_{t_-}}\,|\phi_3\rangle\,\langle\phi_3|\,\hat{1}\,|\phi_2\rangle\,\langle\phi_2|\,\hat{\mathcal{U}}_{\delta_{t_+}}\,|\phi_1\rangle\,\langle\phi_1|\,\hat{\rho}_0\,|\phi_4\rangle\,. \tag{2.3.3}$$

For $\delta_t \ll 1$ we can calculate explicitly the form of

$$\langle\phi_j|\,\hat{\mathcal{U}}_{\delta_{t_\pm}}\,|\phi_{j-1}\rangle = \langle\phi_j|\,e^{\mp i\hat{H}(\hat{b}^\dagger,\hat{b})\delta_t}\,|\phi_{j-1}\rangle = e^{\bar{\phi}_j\phi_{j-1}}e^{\mp iH(\bar{\phi}_j,\phi_{j-1})\delta_t}. \tag{2.3.4}$$

We can then write the partition function as

$$\boxed{Z = \frac{1}{\mathrm{Tr}\{e^{-\beta\hat{H}}\}}\int\prod_{j=1}^{2N} d[\bar{\phi},\phi]\,\exp\left\{i\sum_{j,j'}^{2N}\bar{\phi}_j G^{-1}\phi_{j'}\right\},} \tag{2.3.5}$$

where, using equation (2.3.3), we can see for $N = 2$

$$(iG_{jj'}) = \begin{pmatrix} -1 & 0 & 0 & -\dfrac{\beta}{\bar{\phi}_1\phi_4}H(\bar{\phi}_1,\phi_4) \\ h_- & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & h_+ & -1 \end{pmatrix} \tag{2.3.6}$$

with $h_\pm = 1 \pm H(\bar{\phi}_j,\phi_{j-1})/\bar{\phi}_j\phi_{j-1}$, where $j, j-1$ take the relevant value based on the matrix index.

By taking $N \to \infty$ we define the continuum notation

$$\boxed{Z = \int \mathbf{D}[\bar{\phi},\phi]e^{iS[\bar{\phi},\phi]},} \tag{2.3.7}$$

where $\mathbf{D}[\bar{\phi}(t),\phi(t)]$ is shorthand for $\prod_{j=1}^{2N} d[\bar{\phi}_j,\phi_j]/\,\mathrm{Tr}\{e^{-\beta\hat{H}}\}$. The discrete action takes the form [5]

$$S[\bar{\phi},\phi] = \sum_{j=2}^{2N}\delta t_j\left[i\bar{\phi}_j\frac{\phi_j-\phi_{j-1}}{\delta t_j} - H(\bar{\phi}_j,\phi_{j-1})\right] - \beta H(\bar{\phi}_1,\phi_{2N}). \tag{2.3.8}$$

Taking the continuum limit, this takes the form

$$S[\bar{\phi},\phi] = \int_{\mathcal{C}} dt\,\bar{\phi}(t)\hat{G}^{-1}\phi(t), \tag{2.3.9}$$

This non summed term is omitted from the above and $\hat{G}^{-1}$ depends only on $H(\bar{\phi}_j,\phi_{j-1})$. The final term in equation (2.3.8) contributes to the boundary conditions, and hence can be dropped and reintroduced when required.

2.4 The Konstantinov–Perel' Contour

In the discretized action, equation (2.3.8), the boundary term $-\beta H(\bar{\phi}_1, \phi_{2N})$ carries all of the information about the initial thermal weight $\hat{\rho}_0 = e^{-\beta \hat{H}_0}/Z$, but it sits outside the summed, contour-ordered part of the action. It is natural to ask whether this term can be brought inside the same contour-ordered structure that generates the real-time dynamics, so that the entire object, preparation in equilibrium *and* real-time evolution, is generated by a single ordered exponential. This is possible because the Boltzmann factor is itself an evolution operator in imaginary time. Writing

$$e^{-\beta \hat{H}_0} = e^{-i(z_b - z_a)\hat{H}_0}, \qquad z_b - z_a = -i\beta, \tag{2.4.1}$$

we see that $e^{-\beta \hat{H}_0}$ has exactly the form of $\hat{\mathcal{U}}(z_b, z_a)$ evaluated along a path of length $-i\beta$ in the complex time plane, generated by the free (pre-quench) Hamiltonian $\hat{H}_0$. This motivates extending the Keldysh–Schwinger contour $\gamma_K = \gamma_+ \oplus \gamma_-$ by a third, purely imaginary-time branch γ_M, running from t_0 down to $t_0 - i\beta$:

$$\gamma = \gamma_+ \oplus \gamma_- \oplus \gamma_M. \tag{2.4.2}$$

This three-branch contour is due to Konstantinov and Perel'. The Hamiltonian appearing in the ordered exponential is taken to be the full, time-dependent $\hat{H}(t)$ on the two real-time branches $\gamma_\pm$, and the free Hamiltonian $\hat{H}_0$ on the Matsubara branch γ_M, consistent with the assumption (already used in section 2.2) that interactions are switched on only after t_0. The full evolution operator is then

$$\hat{\mathcal{U}}_\gamma = \hat{T}_\gamma \left[\exp\left\{ -i \int_\gamma dz\, \hat{H}(z) \right\} \right], \tag{2.4.3}$$

where $\hat{T}_\gamma$ orders operators along γ, and where the Matsubara segment contributes a factor equal to $e^{-\beta \hat{H}_0}$: parametrizing $z = t_0 - i\tau$ with $\tau \in [0, \beta]$, so that $dz = -i\, d\tau$,

$$-i \int_{\gamma_M} dz\, \hat{H}_0 = -i(-i) \int_0^\beta d\tau\, \hat{H}_0 = -\beta \hat{H}_0. \tag{2.4.4}$$

In this way the partition function, equation (2.3.2), can be written as a single contour-ordered trace with no density matrix inserted by hand,

$$Z = \frac{\mathrm{Tr}\left\{ \hat{\mathcal{U}}_\gamma \right\}}{\mathrm{Tr}\left\{ e^{-\beta \hat{H}_0} \right\}}, \tag{2.4.5}$$

and the discretized boundary term $-\beta H(\bar{\phi}_1, \phi_{2N})$ of equation (2.3.8) is recovered as simply the contribution of the additional γ_M time slices to the same summed action, rather than as a separate boundary condition. That is, equation (2.3.9) generalizes to

$$S[\phi, \bar{\phi}] = \int_\gamma dt\, \bar{\phi}(z)\, \hat{G}^{-1}\, \phi(z), \tag{2.4.6}$$

with the integral now running over the full three-branch contour γ, and the fields on γ_M playing the role of the Matsubara (imaginary-time) fields familiar from equilibrium field theory. Since γ_+ and γ_- together describe a round trip, $\hat{\mathcal{U}}(t_0, t)\hat{\mathcal{U}}(t, t_0) = \hat{1}$, in the absence of any observable or source inserted along the contour the real-time branches contribute trivially, leaving $\mathrm{Tr}\{\hat{\mathcal{U}}_\gamma\} = \mathrm{Tr}\{e^{-\beta \hat{H}_0}\}$ and hence $Z \equiv 1$ identically. This is the familiar statement that the Keldysh partition function is automatically normalized to unity: it becomes a genuine generating functional only once external sources (or, as in equation (2.3.5) onward, expectation values

of $\phi, \bar{\phi}$) are introduced along the contour. The value of the Konstantinov–Perel' construction is therefore not in the value of Z itself, but in showing that the initial thermal state $\hat{\rho}_0$ need not be treated as data external to the contour formalism; it is generated by the same ordered exponential as the real-time dynamics, simply evaluated along an additional imaginary leg of the same contour.

2.5 Matrix of Green's Functions

By a Gaussian integral [5] one can show

$$\left\langle \phi_j \bar{\phi}_{j'} \right\rangle \equiv \int \mathbf{D}[\bar{\phi}, \phi] \phi_j \bar{\phi}_{j'} \exp\left(i \sum_{k,k'}^{2N} \bar{\phi}_k G^{-1}_{kk'} \phi_{k'} \right) = i G_{jj'}. \tag{2.5.1}$$

There are now four distinct cases of t_j and $t_{j'}$, $t_{j,j'}$ on the forward branch of the contour, t_j on the forward branch and $t_{j'}$ on the backward branch, t_j on the backward branch and $t_{j'}$ on the forward branch, and $t_{j,j'}$ on the backward branch of the contour. This is represented in the matrix below

$$\boxed{G = \begin{pmatrix} G^{\mathcal{C}} & G^{-} \\ G^{+} & \tilde{G}^{\mathcal{C}} \end{pmatrix} \equiv -i \begin{pmatrix} \left\langle T[\phi_j^{+} \bar{\phi}_{j'}^{+}] \right\rangle & \left\langle \phi_j^{-} \bar{\phi}_{j'}^{+} \right\rangle \\ \left\langle \phi_j^{+} \bar{\phi}_{j'}^{-} \right\rangle & \left\langle \tilde{T}[\phi_j^{-} \bar{\phi}_{j'}^{-}] \right\rangle \end{pmatrix},} \tag{2.5.2}$$

where $G^{\mathcal{C}}$ is the contoured ordered Green's functions on the forward branch, G^{-} is the Green's function from the backward to forward branch, G^{+} is the Green's function from the forward to backward branch, and $\tilde{G}^{\mathcal{C}}$ is the contoured ordered Green's function on the backward branch. This is written in the trace notation in equation (2.5.3), and summarised in figure 2.5.1.

$$\boxed{\begin{aligned} -iG^{\mathcal{C}} &= \mathrm{Tr}\Big\{ \rho_0 T_{\mathcal{C}} [b(x, t_j) b^{\dagger}(x', t_{j'})] \Big\} \text{ with } t_j, t_{j'} \in \gamma_{+}, \\ -iG^{-} &= \mathrm{Tr}\Big\{ \rho_0 b(x, t_j) b^{\dagger}(x', t_{j'}) \Big\} \text{ with } t_j \in \gamma_{-},\ t_{j'} \in \gamma_{+}, \\ -iG^{+} &= \mathrm{Tr}\Big\{ \rho_0 b^{\dagger}(x, t_j) b(x', t_{j'}) \Big\} \text{ with } t_j \in \gamma_{+},\ t_{j'} \in \gamma_{-}, \\ -i\tilde{G}^{\mathcal{C}} &= \mathrm{Tr}\Big\{ \rho_0 T_{\mathcal{C}} [b(x, t_j) b^{\dagger}(x', t_{j'})] \Big\} \text{ with } t_j, t_{j'} \in \gamma_{-}. \end{aligned}} \tag{2.5.3}$$

The Green's function in equation (2.5.2) are not all linearly independent. This can be seen in figure 2.5.1 by taking the imaginary part of t to zero. In this way we can re-express the Green's functions as

$$\begin{aligned} G^{\mathcal{C}}(t, t') &= \theta(t - t') G^{-}(t, t') + \theta(t' - t) G^{+}(t, t'), \\ \tilde{G}^{\mathcal{C}}(t, t') &= \theta(t' - t) G^{-}(t, t') + \theta(t - t') G^{+}(t, t'). \end{aligned} \tag{2.5.4}$$

This leads us to the identity

$$G^{\mathcal{C}}(t, t') + \tilde{G}^{\mathcal{C}}(t, t') - G^{+}(t, t') - G^{-}(t, t') = 0 \tag{2.5.5}$$

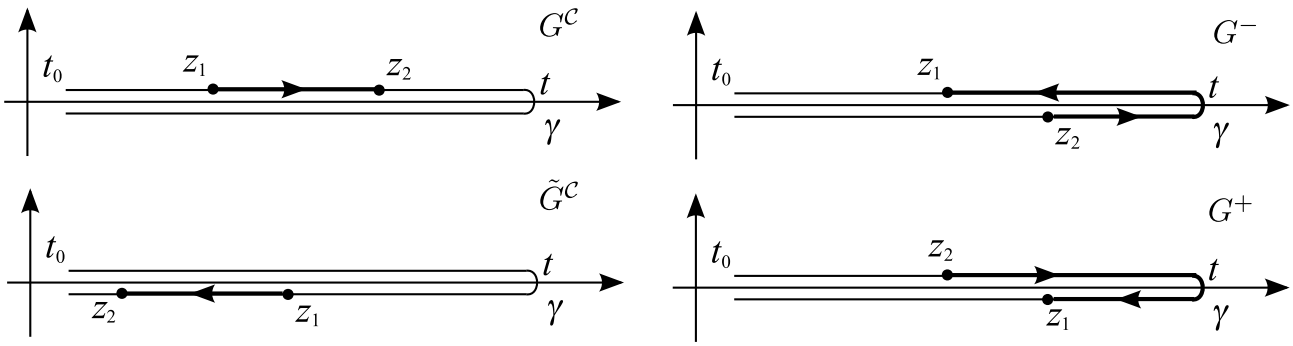


Figure 2.5.1: Graphical representation of the points related by each of the Green's functions in equation (2.5.2). From here it is easy to see how these functions are not all linearly independent, as when the imaginary distance of t is taken to zero, they are no longer distinguishable [6].

3 The Classical Limit

3.1 Keldysh Rotation

By exploiting the relation equation (2.5.5), we can eliminate one of the variables by introducing a new set of fields

$$\phi^{\mathrm{cl}} = \frac{1}{\sqrt{2}}\left(\phi^{+}(t) + \phi^{-}(t)\right), \qquad \phi^{\mathrm{q}} = \frac{1}{\sqrt{2}}\left(\phi^{+}(t) - \phi^{-}(t)\right), \tag{3.1.1}$$

called a **Keldysh rotation** where cl stands for **classical** and q stands for **quantum**, the form of the Green's function matrix, equation (2.5.2), becomes

$$\boxed{\left\langle \phi^{\alpha}(t)\bar{\phi}^{\beta}(t')\right\rangle \equiv iG^{\alpha\beta}(t,t') = i\frac{(\sigma_1+\sigma_3)}{\sqrt{2}}G\frac{(\sigma_1+\sigma_3)}{\sqrt{2}} = \begin{pmatrix} iG^{K}(t,t') & iG^{R}(t,t') \\ iG^{A}(t,t') & 0 \end{pmatrix},} \tag{3.1.2}$$

where (α, β) run over the labels (cl, q), the matrices σ_i are the usual Pauli matrices, the Green's functions are labelled K for **Keldysh**, R for **retarded**, and A for **advanced**. The fourth component is zero by equation (2.5.5).

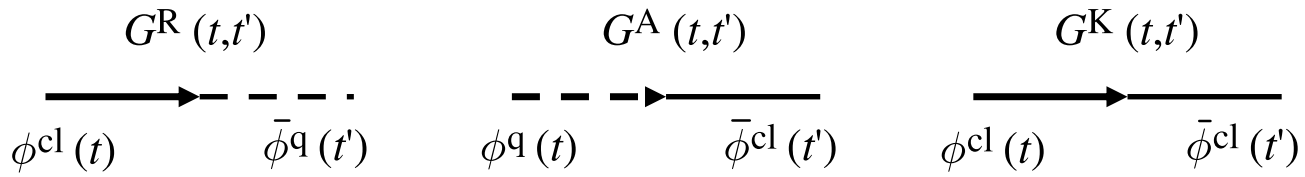


Figure 3.1.1: Diagrammatic representation of the Green's functions equation (3.1.2). The full lines represent the **classical field component** ϕ^{cl} while the dashed line is the **quantum component** ϕ^{q}. The arrows go from annihilation operator to creation operator i.e ϕ^{α} to $\bar{\phi}^{\beta}$. Reproduced from [5].

The **Keldysh action** can then be written as

$$\boxed{S[\vec{\phi}] = \int dt\, \vec{\phi}^{\dagger} G^{-1} \vec{\phi},} \tag{3.1.3}$$

where $\vec{\phi}^{\dagger} = (\bar{\phi}^{\mathrm{cl}}, \bar{\phi}^{\mathrm{q}})$.

3.2 Feynman-Keldysh Action for Real Fields

To convert this to a **classical stochastic system**, one must separate the real and imaginary parts of ϕ. In this way, we write

$$\phi(t) = aX(t) + ibP(t), \tag{3.2.1}$$

such that a, b induce a **canonical transformation**. As an example we show this for the Hamiltonian of a *quantum harmonic oscillator* $\hat{H} = \omega_0 \hat{b}^\dagger \hat{b}$. The transformation equation (3.2.1), takes the form [5]

$$\phi(t) = \frac{1}{\sqrt{2\omega_0}} \left(\omega_0 X(t) + ibP(t)\right), \tag{3.2.2}$$

The Keldysh action takes the form

$$S[X,P] = \frac{1}{2\omega_0} \int_{\mathcal{C}} dt \; \left(\omega_0 X - ibP\right) i\partial_t \left(\omega_0 X + ibP\right) - \omega_0 \left(\omega_0 X + ibP\right)\left(\omega_0 X - ibP\right), \tag{3.2.3}$$

which, after removing terms that are total derivatives, simplifies to

$$S[X,P] = \int_{\mathcal{C}} dt \; \left[P\dot{X} - \frac{1}{2}P^2 - \frac{\omega_0^2}{2}X^2\right]. \tag{3.2.4}$$

Looking at the partition function

$$Z = \int \mathbf{D}[X,P] e^{i\int_{\mathcal{C}} dt \left[P\dot{X} - \frac{1}{2}P^2 - \frac{\omega_0^2}{2}X^2\right]}, \tag{3.2.5}$$

we see that a Gaussian integral can be performed over P, to obtain an equivalent, purely real, action

$$S[X] = \int_{\mathcal{C}} dt \; \left[\frac{1}{2}\dot{X}^2 - \frac{\omega_0^2}{2}X^2\right], \tag{3.2.6}$$

which is the **Feynman-Lagrangian** action of the harmonic oscillator [5]. This generalises to

$$S[X] = \int_{\mathcal{C}} dt \; \left[\frac{1}{2}\dot{X}^2 - V(X)\right], \tag{3.2.7}$$

Using a **Keldysh rotation of real fields**,

$$X^{\mathrm{cl}} = \frac{1}{2}(X^+ + X^-), \tag{3.2.8}$$

$$X^{\mathrm{q}} = \frac{1}{2}(X^+ - X^-). \tag{3.2.9}$$

we can analogously construct the **Feynman-Keldysh action**

$$\boxed{S[X] = \frac{1}{2}\int dt \, \vec{X}^T D^{-1} \vec{X}, \quad D^{-1} = \begin{pmatrix} 0 & [D^{-1}]^A \\ [D^{-1}]^R & [D^{-1}]^K \end{pmatrix},} \tag{3.2.10}$$

where D is the classical Green's function of the system. We are now one step closer to retrieving the classical dynamics of the system.

3.3 The Classical Dissipative Action

Equation (3.2.10) is still fully quantum mechanical. The classical limit is taken by assuming the quantum component is small, $X^{\mathrm{q}} \ll X^{\mathrm{cl}}$. Expanding the different components of D^{-1} to leading order in this limit, the action for a particle coupled to an Ohmic thermal bath reduces to

$$S_{\mathrm{diss}}[X^{\mathrm{cl}}, X^{\mathrm{q}}] = \int_{-\infty}^{+\infty} dt \left[-2X^{\mathrm{q}}\Big(\ddot{X}^{\mathrm{cl}} + \gamma\dot{X}^{\mathrm{cl}} + V'(X^{\mathrm{cl}})\Big) + 4i\gamma T(X^{\mathrm{q}})^2\right], \tag{3.3.1}$$

where γ is a friction coefficient and T the bath temperature. The first term in the action gives the deterministic equation of motion, while the second encodes thermal fluctuations. The limit $\hbar \to 0$ is taken by rescaling all quantum components by $X^{\mathrm{q}} \to \hbar X^{\mathrm{q}}$ and sending $T \to T/\hbar$, so that the action becomes local in time and free of $\hbar$. Quantum fluctuations are thereby discarded while classical thermal fluctuations are fully retained.

One may notice that $S[X^{\mathrm{cl}}, 0] = 0$. This is known as Keldysh symmetry, and it guarantees that the partition function is normalised to one for any value of the classical field. Equivalently, it is a statement of probability conservation. Any classical stochastic action derived from the Keldysh formalism must satisfy this property

4 The Langevin Equation

Since the dissipative action (3.3.1) is at most quadratic in X^{q}, it can be removed by a Hubbard-Stratonovich transformation. This introduces an auxiliary Gaussian field $\xi(t)$ coupled linearly to X^{q}, and performing the required Gaussian integral over X^{q} produces a functional delta function enforcing the equation of motion,

$$\ddot{X}^{\mathrm{cl}} = -\gamma \dot{X}^{\mathrm{cl}} - V'(X^{\mathrm{cl}}) + \xi(t). \tag{4.0.1}$$

This is known as the **Langevin equation**, which is **Newton's law with a friction force** $-\gamma \dot{X}$ and a time-dependent force coming from the auxiliary field $\xi(t)$. Only trajectories satisfying this equation contribute to any observable computed from the classical action.

Since $\xi(t)$ is an auxiliary Gaussian field, it is fully specified by its first two moments. Its mean vanishes, $\xi(t) = 0$, as any nonzero mean can simply be absorbed into the deterministic force $-V'(X^{\mathrm{cl}})$. Its second moment is given by the following:

$$\langle \xi(t)\xi(t') \rangle = \int \mathcal{D}[\xi]\, \xi(t)\xi(t')\, e^{-\frac{1}{4\gamma T}\int dt\, \xi^2} = 2\gamma T\, \delta(t-t'). \tag{4.0.2}$$

Due to this being constant in frequency space , $\xi(t)$ is white noise. Notice that its amplitude is determined by γ and the temperature T. This is a manifestation of the classical fluctuation-dissipation theorem, and ensures that a system driven purely by thermal noise relaxes to the correct Boltzmann distribution.

In the overdamped limit ($\ddot{X}^{\mathrm{cl}}$ negligible compared to $\gamma \dot{X}^{\mathrm{cl}}$), (4.0.1) reduces to the general first-order form

$$\dot{X} = A(X) + b(X)\, \xi(t), \tag{4.0.3}$$

which is often a starting point for stochastic population and ecological models.

5 The Fokker-Planck Equation

The Langevin equation (4.0.3) describes a single stochastic trajectory. To study rare events and escape rates, it is often more convenient to track the full probability distribution $\mathcal{P}(x,t)$, whose evolution is governed by the Fokker-Planck equation. This can be obtained directly from the Keldysh action by working in the stationary-path (MSR) approximation, writing the action of (4.0.3) as

$$S[X,X^{\mathrm{q}}] = \int dt \left[-2X^{\mathrm{q}}\left(\dot{X} - A(X)\right) + 4i(X^{\mathrm{q}})^2 D(X)\right], \tag{5.0.1}$$

where $D(X)$ detemines the strength of the noise, $\langle b(X)\xi(t)\, b(X)\xi(t')\rangle = 2D(X)\delta(t-t')$. If one introduces a momentum $P(t) = 2iX^q(t)$, which is real on stationary trajectories, the action acquires a Hamiltonian structure. With this, we write the **Fokker-Planck Hamiltonian** and the associated equations of motion,

$$H(P,X) = PA(X) + P^2 D(X), \qquad \dot{X} = \partial_P H, \qquad \dot{P} = -\partial_X H, \tag{5.0.2}$$

where $H(P,X)$ is a constant along the trajectory. There are two zero-energy curves: $P=0$, describing noiseless relaxation under the deterministic drift $A(X)$ alone, and $P = -A(X)/D(X)$, describing the time-reversed, fluctuation-driven path back out of a metastable state. These two curves intersect precisely at the fixed points of the noiseless dynamics ($A(X)=0$).

Rather than following individual trajectories, we want the evolution of $\mathcal{P}(X,t)$ itself. This follows by substituting $P \to -\partial_X$ in H, giving the Fokker-Planck equation as a continuity equation,

$$\partial_t \mathcal{P}(X,t) = -\partial_X\Big[A(X)\mathcal{P}(X,t) - \partial_X\big(D(X)\mathcal{P}(X,t)\big)\Big] \;\equiv\; -\partial_X J, \tag{5.0.3}$$

with probability current $J = A\mathcal{P} - \partial_X(D\mathcal{P})$. Conservation of probability is guaranteed by $H(0,X) = 0$.

For additive noise, $D(X) = T$ constant, (5.0.3) can be mapped onto an imaginary-time Schrödinger equation via the canonical transformation $x = X$, $\hat{p} = \hat{P} - V'(X)/(2T)$:

$$\partial_t \Psi(x,t) = -T\left[-\partial_x^2 + W(x)\right]\Psi(x,t), \qquad \Psi(x,t) = e^{V(x)/(2T)}P(x,t), \tag{5.0.4}$$

with the effective potential,

$$W(x) = \frac{[V'(x)]^2}{4T^2} - \frac{V''(x)}{2T}. \tag{5.0.5}$$

The stationary distribution of the original Fokker-Planck problem is then the ground state of the Schrödinger operator $\hat{H}_{\mathrm{Sch}} = T(-\partial_x^2 + W(x))$. In this way, the Fokker-Planck equation can be thought of as a Schrödinger equation corresponding to the Langevin system. When the drift can be written as $A(X) = -\partial_X V$ for some potential $V(X)$, this ground state is simply the Boltzmann distribution $P(X) = Z^{-1}e^{-V(X)/T}$. When this cannot be done, no such closed-form solution exists and one must analyse the metastable states of the system.

6 Classical Stochastic Systems: Noise Induced Stabilisation

To demonstrate some application of the above formalism, particularly to the natural world, we will review a small example of **noise stabilisation** in a stochastic system, described by the Langevin equations equation (4.0.1), with two degrees of freedom X and Y. X can be thought of as the population size, whereas Y is the *sub-population* size, i.e the imbalance between two groups within the main population. This can be used to describe the populations of two competing gene alleles [3]. The Langevin equations read [5][3]

$$\boxed{\dot{X} = X^2 + Y^2 + \xi_X \quad \dot{Y} = -2Y + \xi_Y,} \tag{6.0.1}$$

where $\xi_{X,Y}$ are usual Gaussian *white-noise* that are uncorrelated i.e

$$\begin{aligned} \langle \xi_{X,Y}, \xi_{X,Y} \rangle &= 2T_{X,Y}\delta(t-t'), \\ \langle \xi_X, \xi_Y \rangle &= 0. \end{aligned} \tag{6.0.2}$$

Following the procedure in Parker et al. [3] and the textbook by Kamenev [5], we show that if the variable Y is subject to relatively much stronger noise than X, $T_Y \gg T_X$, the system reaches a metastable state and the system is prevented from escaping. In this way we show that the escape time is exponentially large.

By using the *Fokker-Planck equation*, equation (5.0.3), we can solve for the probability of Y at time t. We can read off the Fokker-Planck equation from equation (6.0.1)

$$\boxed{\partial_t \mathcal{P}(Y,t) = \partial_Y[2Y\mathcal{P}(Y,t)] + T_Y \partial_Y^2 \mathcal{P}(Y,t).} \tag{6.0.3}$$

We impose the absorbing boundary condition $\mathcal{P}(\pm Y_0, t) = 0$ and make the ansatz $\mathcal{P}(Y,t) = e^{-E_Y t}\psi(Y)$. We now have simplified the problem to finding the lowest eigenenergy E_Y. Rewriting the equation and assuming large T_Y noise we have

$$\begin{aligned} -E_Y\psi &= \partial_Y[2Y\psi] + T_Y\partial_Y^2\psi, \\ \Rightarrow -E_Y\psi &\approx T_Y\psi'' \end{aligned} \tag{6.0.4}$$

where it is clear that it takes the form of a particle in a box $\psi \sim \cos\left(\frac{\pi Y}{2Y_0}\right)$ and $E_Y = \frac{\pi^2 T_Y}{4Y_0^2}$ which agrees with Parker et al. [3]. By [3], one expects the probability that the Y motion is confined within the boundaries to be $\exp\{-E_Y(y_0)t_0\}$ while the probability of X being confined within boundaries should be $\exp\{y_0^2/T_X t_0\}$. This implies that the escape time is given by

$$\ln(t_{\text{esc}}) \sim \sqrt{\frac{T_Y}{T_X}}, \tag{6.0.5}$$

which is exponentially large as expected. This means that the noise from the fluctuation of Y induced a metastable state in which the system became bound to. Below is a figure of the numeric solution from Parker et al. of the Fokker-Planck equation of the system described above. The contours and probability current make it clear where the metastable point lies.

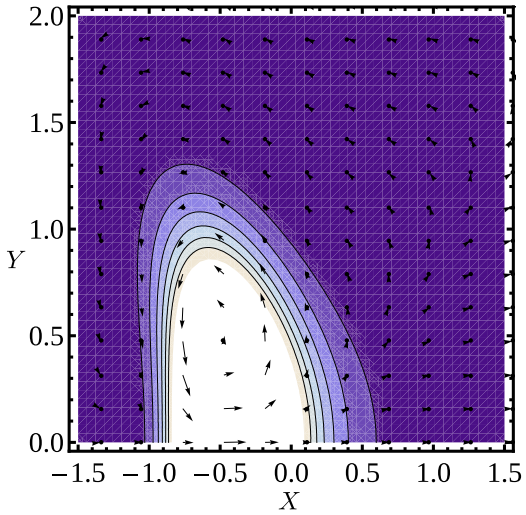


Figure 6.0.1: The contours represent the probability density $\mathcal{P}(X, Y)$ described by equation (5.0.3) and the arrow represent the *probability current*. The system is symmetric around $Y = 0$ and the noise amplitudes are $T_X = 0.05$ and $T_Y = 0.5$. Reproduced from Parker et al. [3].

7 Towards Lattice Population Dynamics: The Bose-Hubbard Model

After developing all of this formalism, the natural next step is to develop a way to use it in some form of model for population dynamics. This will be done by discretizing a space and modeling migration between sites based on some Hamiltonian that can capture immigration and emigration between sites.

A natural microscopic starting point is the Bose-Hubbard model [4],

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1) - \mu \sum_i \hat{n}_i - \sum_i \epsilon_i \hat{n}_i, \tag{7.0.1}$$

where each lattice site i carries a bosonic mode representing a local sub-population. The four terms have direct demographic interpretations: J, the hopping amplitude, models migration between neighbouring sites; U, the on-site interaction, models local competition ($U > 0$) or cooperation ($U < 0$) for resources; μ, the chemical potential, controls the mean population; and ϵ_i, a site-dependent energy offset, models environmental heterogeneity across the lattice (e.g. variation in carrying capacity).

Applying the Keldysh formalism to (7.0.1) would yield Green's functions on the lattice in exactly the same way as for the single-site case, but now carrying a site index. Fitting the resulting parameters (J, U, μ, $\{\epsilon_i\}$) to real demographic data would, in principle, allow this framework to model processes such as human immigration and emigration between regions. This lattice extension is the subject of ongoing work and is not yet reported here.

The *Keldysh Action* for equation (7.0.1) is given by

$$\boxed{S[\bar{\phi}, \phi] = \sum_i \int dt \left[\bar{\phi}_i i \partial_t \phi_i + J \sum_{\langle p,q \rangle} \bar{\phi}_p \phi_q - \frac{U}{2} \bar{\phi}_i \phi_i (\bar{\phi}_i \phi_i - 1) - \mu \bar{\phi}_i \phi_i - \epsilon_i \bar{\phi}_i \phi_i \right].} \tag{7.0.2}$$

To obtain the *Feynman-Keldysh* action of the system is a non-trivial task as the hopping term has a complex structure of pairwise products. Furthermore the self-repulsion term U, is quartic, meaning a canonical transformation of the form equation (3.2.1) is not possible.

Further work will include, justifications to simplify this action, one being the dropping of the quartic term, as the particular type of self interaction it describes is best for condensed matter systems. We will then retrieve an action in terms of a real field in which, after finding the Langevin equations, we will run a numerical simulation of the model for a computationally convenient number of lattice points.

8 Conclusions

In this report, we have systematically demonstrated the utility of the non-equilibrium Keldysh path-integral formalism as a foundational framework for classical stochastic systems. By establishing a rigorous field-theoretic baseline using bosonic coherent states and the Keldysh contour, we successfully mapped the quantum partition function into a continuous action. Taking the semi-classical limit ($X^{\mathrm{cl}} \gg X^{\mathrm{q}}$) stripped the system of purely quantum fluctuations while fully retaining classical thermal fluctuations, ultimately establishing that the stochastic forcing of the Langevin equation is a natural consequence of the field's underlying quantum components rather than an arbitrary stochastic addition.

Through the stationary-path approximation of the MSR action, this framework effortlessly transitions to a Fokker-Planck description, exposing an elegant underlying Hamiltonian structure. The dual zero-energy paths of this effective Hamiltonian provide a powerful geometric interpretation of dynamical behaviour, separating deterministic relaxation from rare, fluctuation-driven transitions out of metastable states. We applied this machinery to a two-variable population model to reproduce the phenomenon of noise-induced stabilization. This demonstrated that strong stochastic fluctuations in an auxiliary sub-population can exponentially prolong the lifetime of an otherwise unstable global population state, with implications for ecological modelling and genetic allele dynamics.

The ultimate objective of this framework is its application to human population dynamics via an adapted lattice Bose-Hubbard model. While the microscopic model provides intuitive demographic parallels, where hopping (J) governs spatial migration, chemical potential (μ) regulates global carrying capacity, and environmental heterogeneity (such as mountainous or rough terrain) is encoded via site-dependent offsets (ϵ_i), the quartic on-site interaction (U) introduces severe analytical non-linearities that resist standard real-field canonical transformations. Ongoing and future work will focus on introducing well-justified simplifications, primarily removing or linearising the quartic self-repulsion term to capture macroscale human migration trends cleanly. The resulting quadratic real action will then be used to derive explicit spatial Langevin equations, providing the mathematical architecture for robust numerical simulations across a computationally viable lattice network.

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