Keldysh Field Theory for Lattice Population Dynamics

Daniel Barron & O. Daniel O'Carroll
Supervisor: Prof. Karen Wiltshire

Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Introduction

The study of stochastic systems lends itself to applications in population dynamics for its robust treatment of non-equilibrium systems. In nature, it has been observed that systems can fall into **metastable states** which are best described by the type of dynamics outlined above [1] [2]. In this project, we start by reviewing the **Keldysh-Schwinger formalism** of statistical systems out of equilibrium. We show how this formalism, while quantum by nature, encodes classical stochastic dynamics of fluctuating systems. By taking the real part of the field in the **Keldysh action**, one can obtain a **Langevin equation** via a **Hubbard-Stratonovich transformation**, and, by extension, a **Fokker-Planck equation**. We review how these methods and techniques are used to describe systems in nature, as outlined above, and plan to explore their application to **human population dynamics**.

Green's Functions

For **non-equilibrium** systems, we evolve the system through imaginary time to compute real observables. We do this in a **discrete way**, taking N discrete time steps of length δ_t as shown in figure 1.

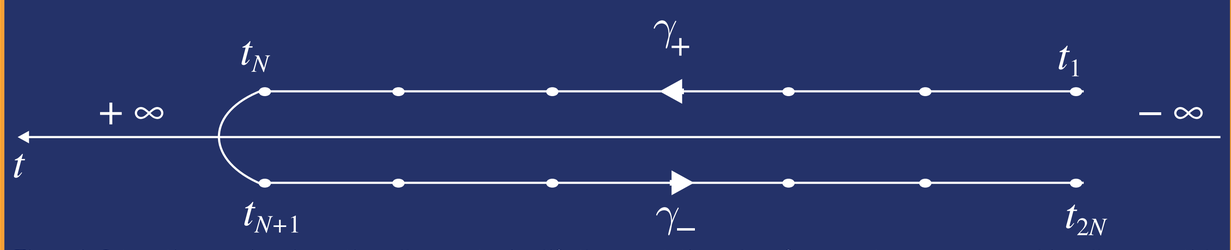


Figure 1: Integrating over imaginary time allows us to simplify the expectation value of an operator, as we can combine all exponentials in the expectation value. The contour is called the **Keldysh contour** and is denoted by $\gamma_K = \gamma_+ \oplus \gamma_-$. Reproduced from [3].

For a basis of **bosonic coherent states** $\phi_n\bar{\phi}_n$, where n is the time step, the **partition function** takes the form

$$Z = \frac{\mathrm{Tr}\left\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}_0\right\}}{\mathrm{Tr}\{\hat{\rho}_0\}} = \frac{1}{\mathrm{Tr}\{\hat{\rho}_0\}}\int\prod_{j=1}^{2N} d[\bar{\phi}_j,\phi_j]\exp\left(i\sum_{j,j'}^{2N}\bar{\phi}_j G^{-1}_{jj'}\phi_{j'}\right), \tag{1}$$

where $\hat{\mathcal{U}}_{\mathcal{C}}$ is the **evolution operator** across the time contour $\mathcal{C}$, $\hat{\rho}_0$ is the **initial density operator**, and $G^{-1}_{jj'}$ is the **inverse two-point function** between time steps j and j'.

By the properties of **Gaussian integrals** we find [4][3]

$$\left\langle\phi_j\bar{\phi}_{j'}\right\rangle \equiv \int\mathbf{D}[\bar{\phi},\phi]\phi_j\bar{\phi}_{j'}\exp\left(i\sum_{j,j'}^{2N}G^{-1}_{j,j'}\right) = \mathrm{Tr}\left\{\hat{\rho}_0 T_{\mathcal{C}}[b(x,t_j)b^\dagger(x',t_{j'})]\right\} = iG_{jj'}, \tag{2}$$

where we now use the **continuum notation** $\int\mathbf{D}[\bar{\phi},\phi] \equiv \frac{1}{\mathrm{Tr}\{\hat{\rho}_0\}}\int\prod_{j=1}^{2N}d[\bar{\phi}_j,\phi_j]$, and $T_{\mathcal{C}}$ is the **time ordering operator along the contour**. This gives rise to a matrix of four Green's functions depending on whether the points $t_{j,j'}$ are on the forward or backward branch of the contour $\mathcal{C}$, and whether t_j precedes or succeeds $t_{j'}$. By relabelling the fields ϕ to $\phi_\pm$ for the forward and backwards parts of the contour respectively.

$$\begin{aligned}
-iG^{\mathcal{C}} &= \mathrm{Tr}\left\{\rho_0 T_{\mathcal{C}}[b(x,t_j)b^\dagger(x',t_{j'})]\right\} \text{ with } t_j,t_{j'}\in\gamma_+,\\
-iG^{-} &= \mathrm{Tr}\left\{\rho_0 T_{\mathcal{C}}[b(x,t_j)b^\dagger(x',t_{j'})]\right\} \text{ with } t_j\in\gamma_-,\ t_{j'}\in\gamma_+,\\
-iG^{+} &= \mathrm{Tr}\left\{\rho_0 T_{\mathcal{C}}[b(x,t_j)b^\dagger(x',t_{j'})]\right\} \text{ with } t_j\in\gamma_+,\ t_{j'}\in\gamma_-,\\
-i\tilde{G}^{\mathcal{C}} &= \mathrm{Tr}\left\{\rho_0 T_{\mathcal{C}}[b(x,t_j)b^\dagger(x',t_{j'})]\right\} \text{ with } t_j,t_{j'}\in\gamma_-.
\end{aligned} \tag{3}$$

$$G = \begin{pmatrix} G^{\mathcal{C}} & G^{-} \\ G^{+} & \tilde{G}^{\mathcal{C}}\end{pmatrix} \equiv -i\begin{pmatrix}\left\langle T[\phi_j^+\bar{\phi}_{j'}^+]\right\rangle & \left\langle\phi_j^-\bar{\phi}_{j'}^+\right\rangle \\ \left\langle\phi_j^+\bar{\phi}_{j'}^-\right\rangle & \left\langle\tilde{T}[\phi_j^-\bar{\phi}_{j'}^-]\right\rangle\end{pmatrix} \tag{4}$$

The **time-ordered** Green's functions are related by

$$\begin{gathered}G^{\mathcal{C}} = \theta(t-t')G^- + \theta(t'-t)G^+, \quad \tilde{G}^{\mathcal{C}} = \theta(t-t')G^+ + \theta(t'-t)G^-,\\ \Rightarrow G^{\mathcal{C}} + \tilde{G}^{\mathcal{C}} - G^+ - G^- = 0.\end{gathered} \tag{5}$$

Keldysh Rotation

By introducing a new set of fields

$$\phi^{\mathrm{cl}} = \frac{1}{\sqrt{2}}\left(\phi^+(t)+\phi^-(t)\right), \qquad \phi^{\mathrm{q}} = \frac{1}{\sqrt{2}}\left(\phi^+(t)-\phi^-(t)\right), \tag{6}$$

called a **Keldysh rotation** where cl stands for **classical** and q stands for **quantum**, the form of the Green's function matrix equation (4) becomes

$$\left\langle\phi^\alpha(t)\bar{\phi}^\beta(t')\right\rangle \equiv iG^{\alpha\beta}(t,t') = \frac{(\sigma_1+\sigma_3)}{\sqrt{2}}G\frac{(\sigma_1+\sigma_3)}{\sqrt{2}} = \begin{pmatrix} iG^K(t,t') & iG^R(t,t') \\ iG^A(t,t') & 0\end{pmatrix}, \tag{7}$$

where (α,β) run over the labels (cl, q), the matrices σ_i are the usual Pauli matrices, the Green's functions are labelled K for **Keldysh**, R for **retarded**, and A for **advanced**. The fourth component is zero by equation (5).

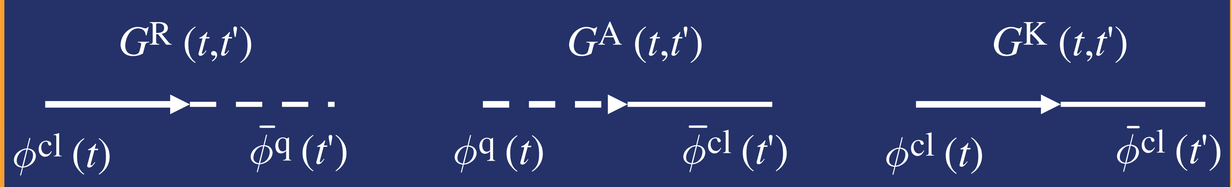


Figure 2: Diagrammatic representation of the Green's functions equation (7). The full lines represent the **classical field component** ϕ^{cl} while the dashed line is the **quantum component** ϕ^{q}. The arrows go from annihilation operator to creation operator i.e ϕ^α to $\bar{\phi}^\beta$. Reproduced from [3].

The **Keldysh action** can then be written as

$$S[\vec{\phi}] = \int dt\vec{\phi}^\dagger G^{-1}\vec{\phi}, \tag{8}$$

where $\vec{\phi}^\dagger = (\bar{\phi}^{\mathrm{cl}},\bar{\phi}^{\mathrm{q}})$. To convert this to a **classical stochastic system**, one must separate the real and imaginary parts of ϕ. In this way, we write $\phi(t) = aX(t)+bP(t)$, such that a,b induce a **canonical transformation**. We can then integrate out P from the partition function to find an equivalent action and obtain a function only of the real field X. Using a **Keldysh rotation of real fields**,

$$X^{\mathrm{cl}} = \frac{1}{2}(X^+ + X^-), \tag{9}$$

$$X^{\mathrm{q}} = \frac{1}{2}(X^+ - X^-), \tag{10}$$

we can analogously construct the **Feynman-Keldysh action**

$$S[X] = \frac{1}{2}\int dt\vec{X}^T D^{-1}\vec{X}, \quad D^{-1} = \begin{pmatrix}0 & [D^{-1}]^A \\ [D^{-1}]^R & [D^{-1}]^K\end{pmatrix}. \tag{11}$$

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Classical Dissipative Action

Assuming that the **quantum fluctuations** are small ($X^{\mathrm{q}} \ll X^{\mathrm{cl}}$) the **Feyman-Keldysh action** [3] reduces to a classical dissipative action of a particle connected to an **Ohmic bath**,

$$S_{\mathrm{diss}}[X^{\mathrm{cl}},X^{\mathrm{q}}] = \int_{-\infty}^{+\infty}\mathrm{d}t\,\left[-2X^{\mathrm{q}}\left(\ddot{X}^{\mathrm{cl}}+\gamma\dot{X}^{\mathrm{cl}}+V'(X^{\mathrm{cl}})\right)+4i\gamma T(X^{\mathrm{q}})^2\right]. \tag{12}$$

The first term in the action gives the deterministic equation of motion, while the second encodes thermal fluctuations. The limit $\hbar\to 0$ is taken by rescaling all quantum components by $X^{\mathrm{q}}\to\hbar X^{\mathrm{q}}$ and sending $T\to T/\hbar$, so that the action becomes local in time and free of $\hbar$. Quantum fluctuations are thereby discarded while classical thermal fluctuations are fully retained. The property that $S[X^{\mathrm{cl}},0]=0$ is known as Keldysh symmetry, and is crucial to the conservation of probability.

Langevin Equation

Applying the **Hubbard-Stratonovich transformation** to the quadratic X^{q} term in the action [3] introduces an auxiliary field $\xi(t)$ and enforces the equation of motion as a functional delta function. Therefore, only trajectories that satisfy

$$\ddot{X}^{\mathrm{cl}} = -\gamma\dot{X}^{\mathrm{cl}} - V'(X^{\mathrm{cl}}) + \xi(t) \tag{13}$$

contribute to any observable. This is known as the **Langevin equation**, which is **Newton's law with a friction force** $-\gamma\dot{X}$ and a time-dependent force coming from the auxiliary field $\xi(t)$. This is a random force that follows Gaussian statistics, hence, only its first and second irreducible moments must be specified. In our case, $\langle\xi(t)\rangle = 0$, since any non-zero first moment can be viewed as a part of the deterministic force $-V'(X^{\mathrm{cl}})$. The second moment is then given by,

$$\langle\xi(t)\xi(t')\rangle = \int\mathbf{D}[\xi]\,\xi(t)\xi(t')\,e^{-\frac{1}{4\gamma T}\int\mathrm{d}t\,\xi^2} = 2\gamma T\delta(t-t'). \tag{14}$$

Since in the frequency representation the right-hand side of this equation is constant, the corresponding force is often referred to as white noise. The noise amplitude being set by the friction coefficient γ and the temperature T is a manifestation of the **fluctuation-dissipation theorem** (FDT) in its classical limit.

In the overdamped limit, this reduces to something of the form $\dot{X} = A(X)+b(X)\xi(t)$, which is the standard starting point for stochastic population models and ecological dynamics. The two variable system of [5] is of this form, with the equations of motion being,

$$\begin{aligned}
\dot{X} &= -V'(X) - Y^2 + \xi_X(t),\\
\dot{Y} &= -2Y + \xi_Y(t),\\
\langle\xi_{X(Y)}(t)\xi_{X(Y)}(t')\rangle &= 2T_{X(Y)}\delta(t-t'),\\
V(X) &= -X^3/3 - \epsilon X.
\end{aligned} \tag{15}$$

Here, X represents the total population size, while Y is some additional degree of freedom, for example, it could be due to the imbalance between the numbers of two competing gene alleles. It should also be mentioned that ϵ is the bifurcation parameter.

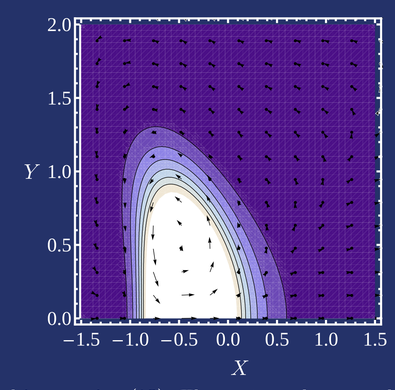


Figure 3: The dynamics of the system outlined in equation (15). The contours here are the probability $\mathcal{P}$, which can be calculated using the **Fokker-Planck** equation as outlined below. The arrows represent the probability current density. The system is symmetric around $y=0$, and the values of T_X and T_Y are 0.05 and 0.5 respectively. Reproduced from [5].

Fokker-Planck Equation

To analyse rare events and escape rates, we work in the stationary path approximation of the **MSR action**,

$$S[X,X^{\mathrm{q}}] = \int\mathrm{d}t\,\left(-2X^{\mathrm{q}}\left(\partial_t^{\mathrm{R}}X - A(X)\right)+4i(X^{\mathrm{q}})^2D(X)\right). \tag{16}$$

If one introduces a momentum $P(t) = 2iX^q(t)$, which is real on stationary trajectories, the action acquires a Hamiltonian structure. With this, we write the **Fokker-Planck Hamiltonian** and the associated equations of motion,

$$H(P,X) = PA(X) + P^2D(X), \qquad \dot{X} = \partial_P H, \qquad \dot{P} = -\partial_X H, \tag{17}$$

where $H(P,X)$ is a constant along the trajectory. The two zero-energy curves $P=0$ (noiseless relaxation) and $P=-A(X)/D(X)$ (fluctuation path, time-reversed relaxation) intersect where $A(X)=0$, namely at the fixed points of the noiseless dynamics.

Instead of tracking individual trajectories, the Fokker-Planck equation governs the full probability distribution $\mathcal{P}(X,t)$. The corresponding restricted partition function $\mathcal{P}(X,t) = Z|_{X(t)=X}$ is proportional to the probability of finding the system at the point X at time t. The Fokker-Planck equation has the form $\partial_t\mathcal{P} = \hat{H}\mathcal{P}$ and is obtained by subtituting $P\to-\partial_X$,

$$\partial_t\mathcal{P}(X,t) = -\partial_X\left[A(X)\mathcal{P}(X,t) - \partial_X[D(X)\mathcal{P}(X,t)]\right]. \tag{18}$$

This is a continuity equation $\partial_t\mathcal{P}+\partial_X J = 0$, with probability current $J = AP - \partial_X[DP]$. Conservation of probability is guaranteed by $H(0,X)=0$. For additive noise $D(X)=T$, the Fokker-Planck Hamiltonian may be transformed into an **imaginary-time Schrödinger equation** by the canonical transformation $x=X$ and $\hat{p}=\hat{P}-V'(X)/(2T)$,

$$\partial_t\Psi(x,t) = -T[-\partial_x^2 + W(x)]\Psi(x,t), \tag{19}$$

where the effective potential $W(x)$ and wavefunction $\Psi(x,t)$ are,

$$W(x) = \frac{[V'(x)]^2}{4T^2} - \frac{V''(x)}{2T}; \qquad \Psi(x,t) = e^{V(x)/(2T)}\mathcal{P}(x,t). \tag{20}$$

The stationary distribution of this system corresponds to the ground state of this Schrödinger operator. This ground state gives a Boltzmann distribution $\mathcal{P}(X) = Z^{-1}e^{-V(X)/T}$. However, when the drift $A(X)$ cannot be written as $-\partial_X V$ for any potential $V(X)$, no such solution exists.

Key Results & Future Work

- Derived stochastic dynamics from the Keldysh formulism of a non-equilibrium quantum system.
- Showed how noise arises from the quantum component of the Keldysh field via a Hubbard-Stratonovich calculation.
- Studied how stochastic methods, Langevin & Fokker-Planck equations, can be applied to real-world stochastic systems.
- Plan to use **Bose-Hubbard model**, which describes bosons on a lattice, and retrieve stochastic differential equations, to model the effects of human immigration and emigration on local populations.