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Dimer models as models of Majorana fermions

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School of Theoretical Physics

Overview

1. Background
2. The Hexagonal Lattice
3. A Non-Bipartite Hexagonal Lattice
4. The Triangular Lattice
5. A Bipartite Hexagonal Lattice with a Defect
6. Conclusion

Background

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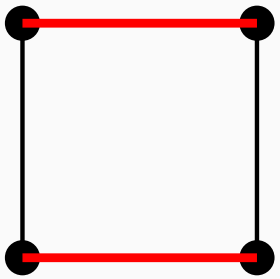

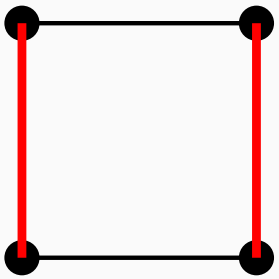

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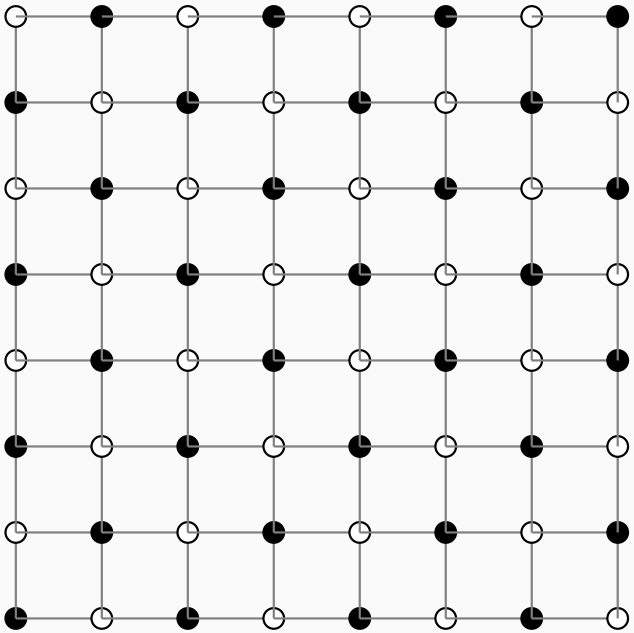
- A dimer covering is a way of arranging pairs of adjacent vertices (or sites) on a graph or lattice such that each vertex is covered by exactly one dimer, with no overlaps.

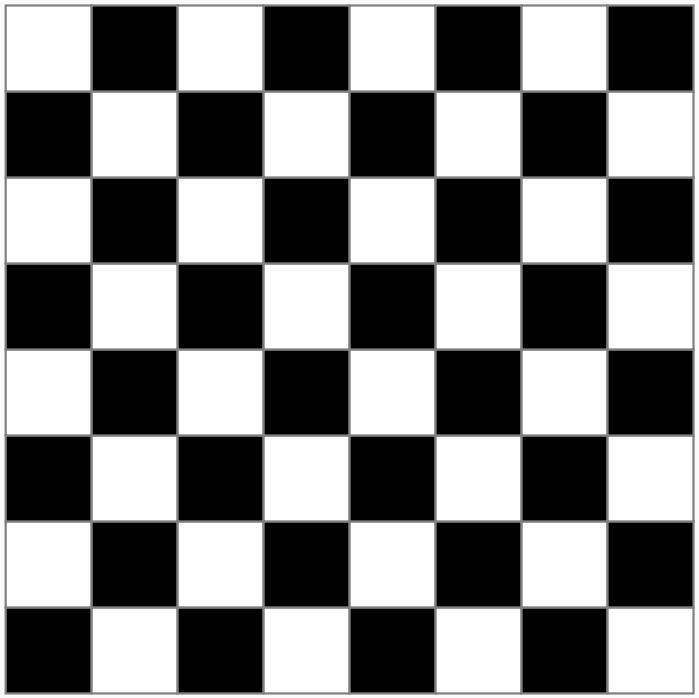


- There is a one-to-one mapping between this graph picture and its dual, obtained by replacing vertices with faces and crossing edges perpendicularly.

Background: Visualisation of Square Graph and its Dual



Bipartite Square Grid Graph

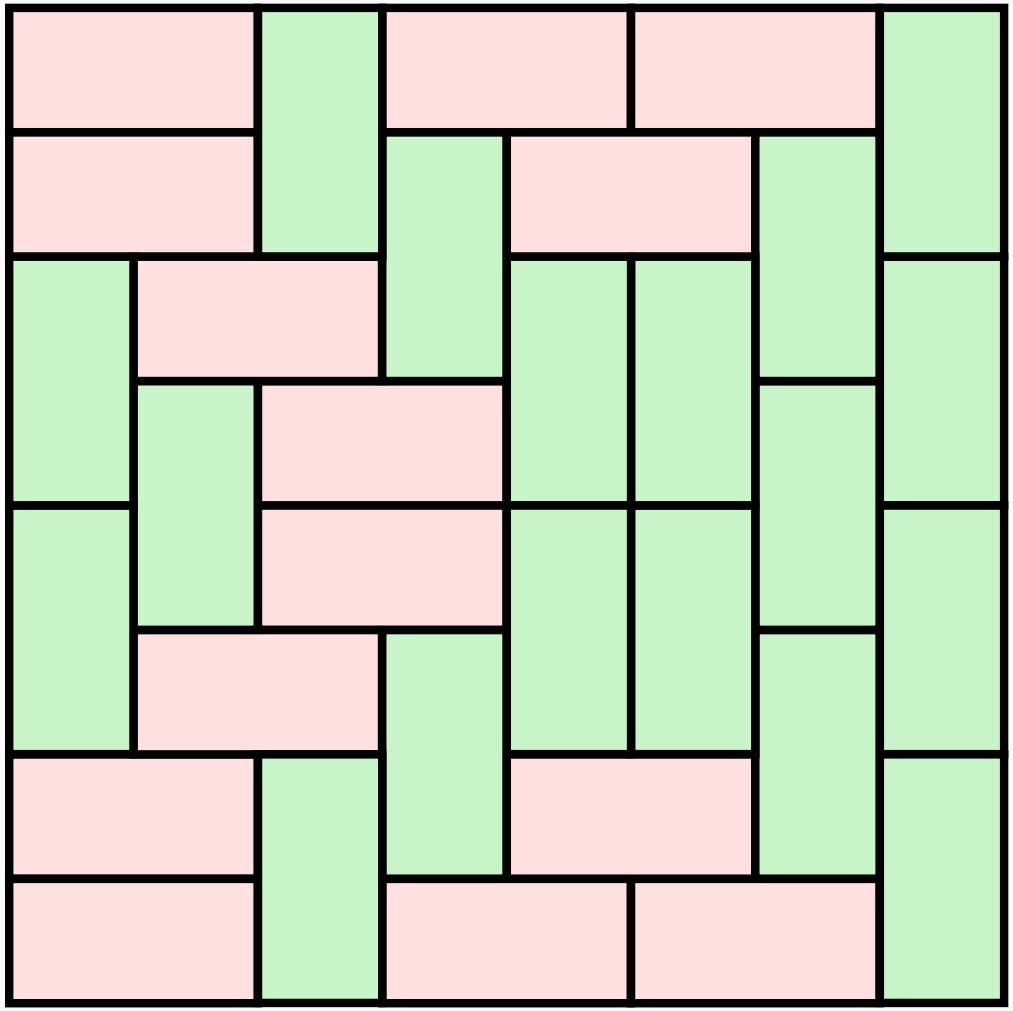


Dual Graph: Chessboard Tiling

Background: Domino Tiling of a Chessboard [6]

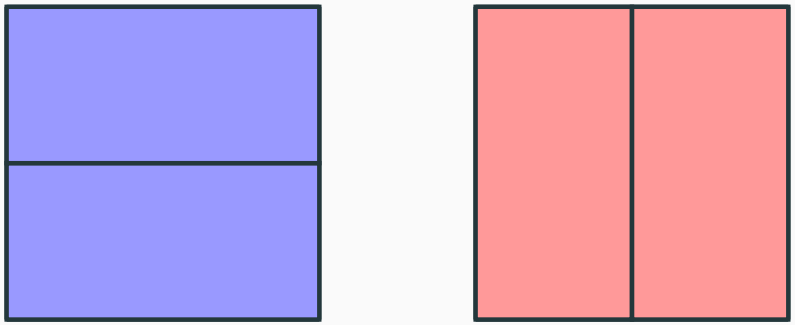
Question

How many different covers are there for an 8×8 chessboard?



Background: Counting Dimer Coverings

- Simple enough to count for a 2×2 chessboard.



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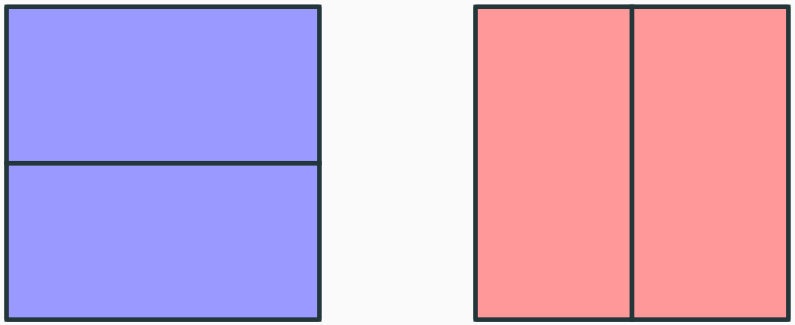

- However, the number of different coverings gets very large very fast [2].

Size of board	No. of coverings
2×2	2
4×4	36
6×6	6,728
8×8	12,988,816
10×10	258,584,046,368

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- As was shown before, these dimer configurations become difficult to count quickly even at small lattice sizes, which is why we use Kasteleyn's Theorem to compute the partition function.

Background: Kasteleyn's Theorem

- The key is to assign a *Kasteleyn orientation* to each edge of the graph Γ so that each face of the graph has an odd number of edges oriented clockwise.

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- For graphs embedded on closed oriented surfaces of genus g, such as the torus, the full partition function is given by [1]:

$$\mathcal{Z}(\Gamma) = \frac{1}{2^g} \sum_{\vec{u} \in S(\Sigma)} \mathrm{Arf}(\vec{u}) \mathrm{Pf}(K_{\vec{u}}) \tag{4}$$

Background: Why do we care?

- At $\beta = 0$, all bond activities become 1, so

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- One can use this to compute combinatoric quantities such as the number of domino tilings on an $M \times N$ rectangle:

$$\prod_{j=1}^{\lceil M/2 \rceil} \prod_{k=1}^{\lceil N/2 \rceil} \left[4\cos^2\left(\frac{\pi j}{M+1}\right) + 4\cos^2\left(\frac{\pi k}{N+1}\right) \right] \tag{6}$$

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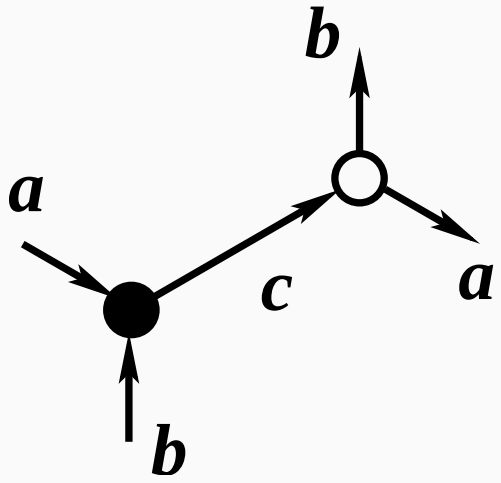
- These can then be studied in further detail to observe and teach us about phase transitions, which is always a topic of interest to physicists.

The Hexagonal Lattice

Hexagonal Lattice: Setup

- For the simplest case of the periodic hexagonal lattice, there are three different types of bonds, denoted by activities a, b and c. These are shown in the diagram below of a fundamental tile of this lattice [4].



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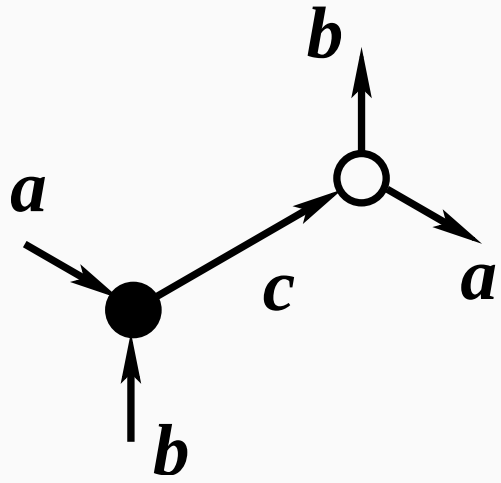

- The partition function is then simply the following:

$$\mathcal{Z} = \sum_{\text{coverings}} a^{N_a} b^{N_b} c^{N_c} \tag{7}$$

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- This lattice is known as a bipartite lattice, which is one that can be split into black and white vertices, where vertices of the same colour never share an edge.

Hexagonal Lattice: Setup

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- The Kasteleyn matrix of a bipartite lattice can be written in a special form. For bipartite graphs with $2n$ vertices, K takes the form:

$$K = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}, \tag{8}$$

where A is an $n \times n$ matrix mapping from black to white vertices.

Hexagonal Lattice: Setup

- The matrix A for a simple 2×2 fundamental tile lattice with doubly periodic boundary conditions looks like the following, for example:

$$A = \begin{pmatrix} c & -a & -b & 0 \\ -a & c & 0 & -b \\ -b & 0 & c & -a \\ 0 & -b & -a & c \end{pmatrix} \tag{9}$$

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- Summing over the different boundary conditions in both directions, the explicit form of Equation (4) in this case is:

$$\mathcal{Z} = \frac{1}{2}\left(- \det \begin{pmatrix} c & -a & -b & 0 \\ -a & c & 0 & -b \\ -b & 0 & c & -a \\ 0 & -b & -a & c \end{pmatrix} + \det \begin{pmatrix} c & a & -b & 0 \\ -a & c & 0 & -b \\ -b & 0 & c & a \\ 0 & -b & -a & c \end{pmatrix} + \right. \tag{10}$$

$$\left. + \det \begin{pmatrix} c & -a & b & 0 \\ -a & c & 0 & b \\ -b & 0 & c & -a \\ 0 & -b & -a & c \end{pmatrix} + \det \begin{pmatrix} c & a & b & 0 \\ -a & c & 0 & b \\ -b & 0 & c & a \\ 0 & -b & -a & c \end{pmatrix} \right) \tag{11}$$

$$= a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 = (a^2 + b^2 + c^2)^2 \tag{12}$$

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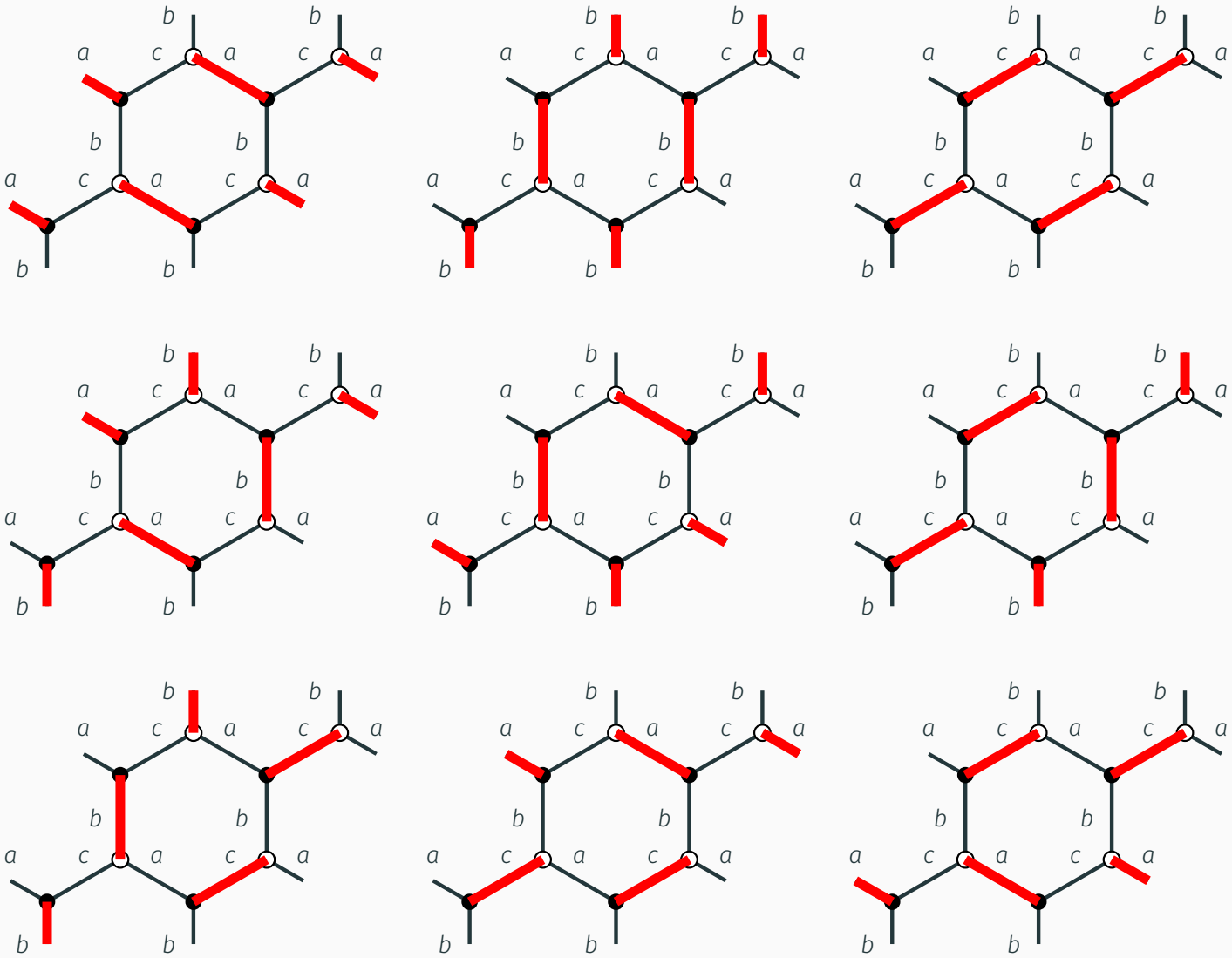
Hexagonal Lattice: 2×2 Fundamental Tile Dimer configurations



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Hexagonal Lattice: The Spectral Polynomial

- In general, the doubly periodic A matrix can be written in the form:

$$A_{k_1,k_2;l_1l_2} = c\delta_{k_1,k_2}\delta_{l_1,l_2} - a\delta_{k_1,k_2+1}\delta_{l_1,l_2} - b\delta_{k_1k_2}\delta_{l_1,l_2+1} \tag{13}$$

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- This matrix can be diagonalised using the eigenvectors $z^k w^{-l}$, one can see

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- Hence, the eigenvalues are given by what is known as the spectral polynomial,

$$p^{Hex}(z,w) = c - a/z - bw. \tag{15}$$

Hexagonal Lattice: Free Energy in the Thermodynamic Limit

- Now, using the fact that these indices are periodic like the lattice, $z^N = 1$ and $w^M = 1$ for an $N \times M$ fundamental tile lattice. Hence, these can be written as simply

$$z = e^{2\pi i n/N} = e^{i\theta}, \quad w = e^{2\pi i m/M} = e^{i\phi}, \tag{16}$$

where $\theta = \frac{2\pi n}{N}$ and $\phi = \frac{2\pi m}{M}$.

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- The determinant of A is now just a product over the spectral polynomials:

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- We can now compute the "Free energy per dimer" $W = \frac{\ln \mathcal{Z}}{NM}$ in the thermodynamic limit:

$$W = \iint \frac{d\theta d\phi}{4\pi^2} \ln(c - a e^{-i\theta} - b e^{i\phi}) \tag{18}$$

Hexagonal Lattice: Phase Transitions

- W can be used to compute important thermodynamic properties of the system,

$$\text{Helmholtz free energy per dimer: } \mathcal{F} = -\frac{1}{\beta} W \tag{19}$$

$$\text{Internal energy per dimer: } U = -\frac{\partial W}{\partial \beta} \tag{20}$$

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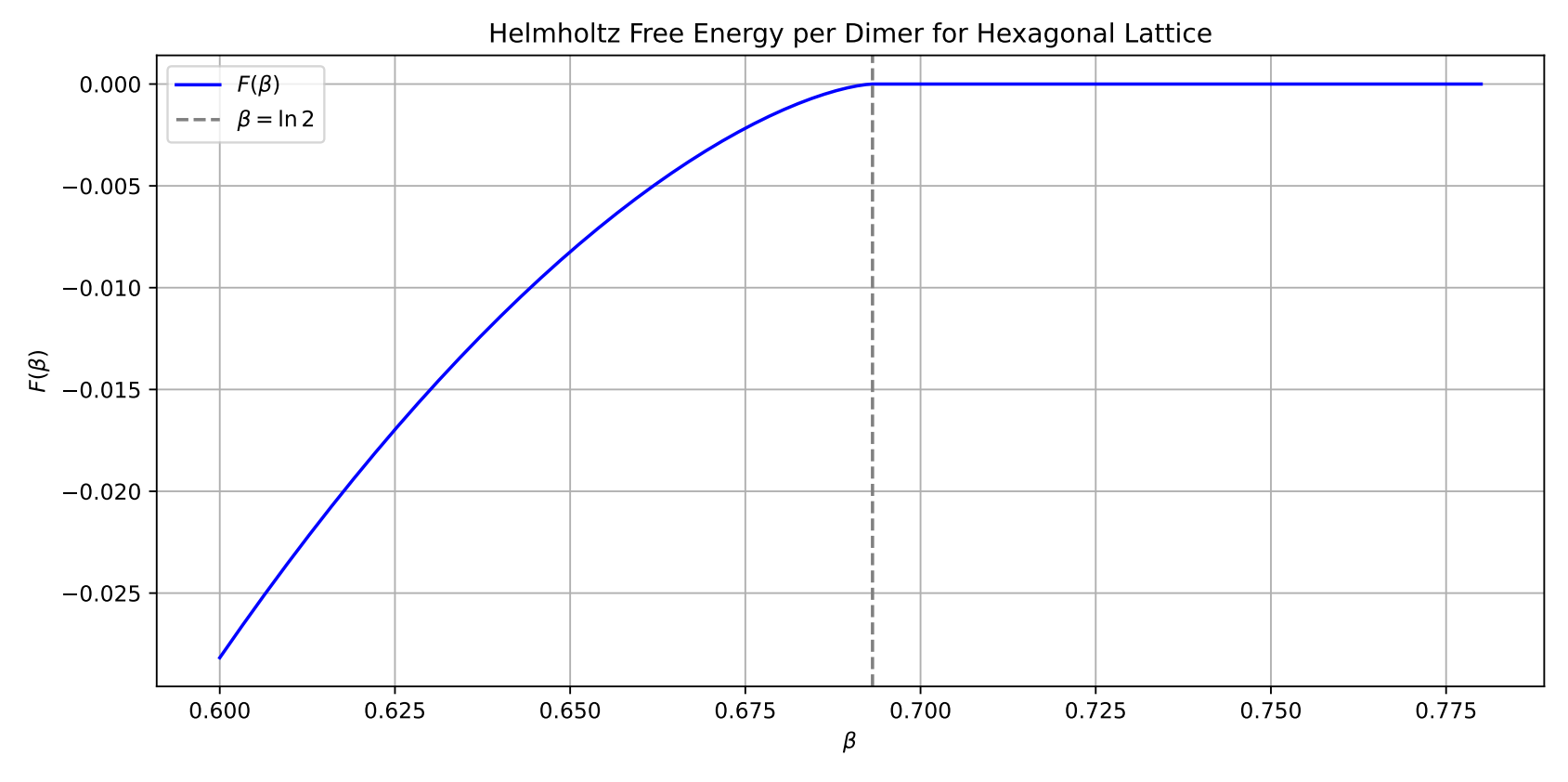
- Hence, at the point where one weight is equal to the sum of the other two, there is a phase transition.
- We can see this by plotting these quantities for $c = 1$ and $a = b = e^{-\beta}$.

Hexagonal Lattice: Thermodynamic Quantities



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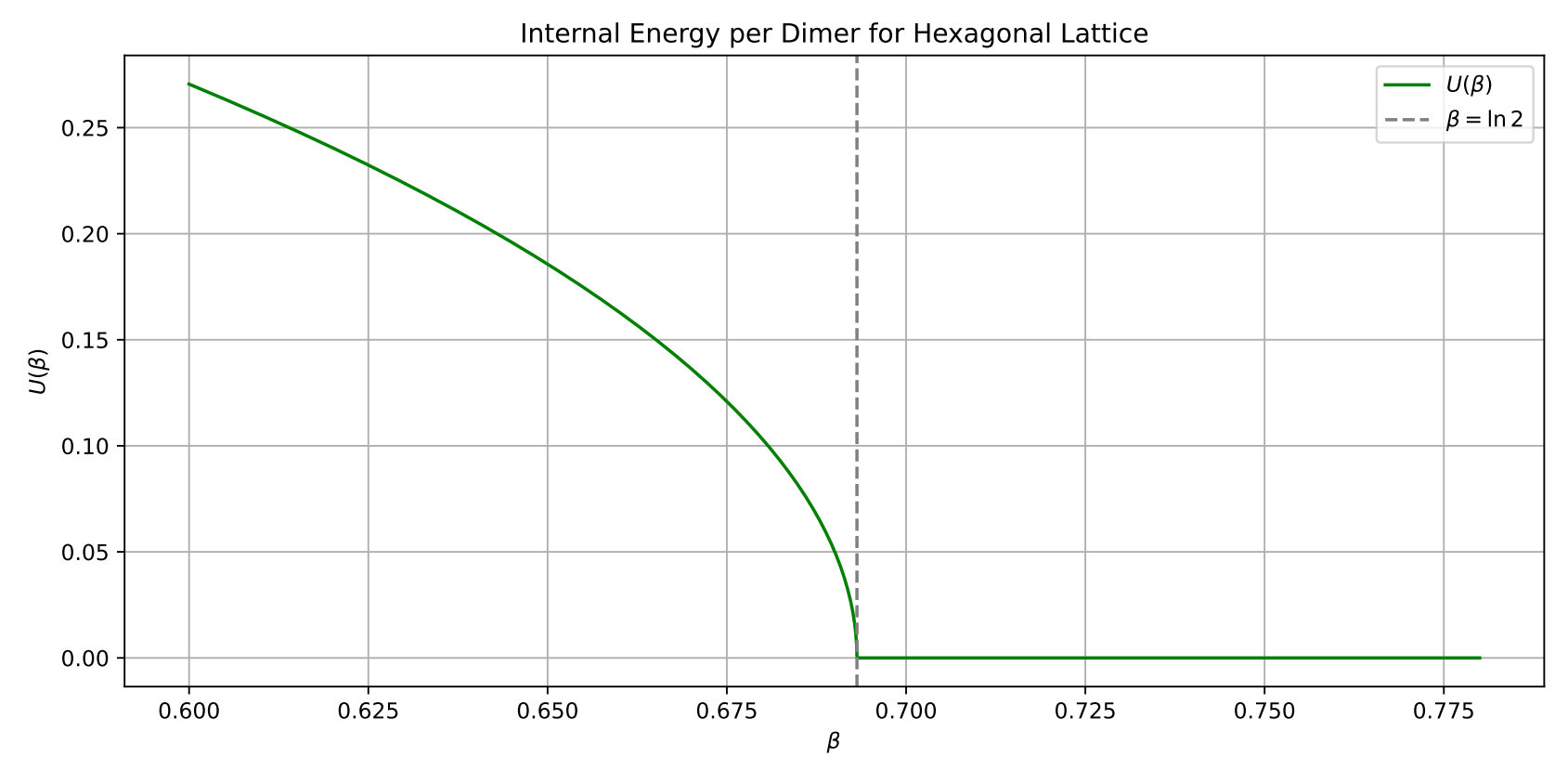

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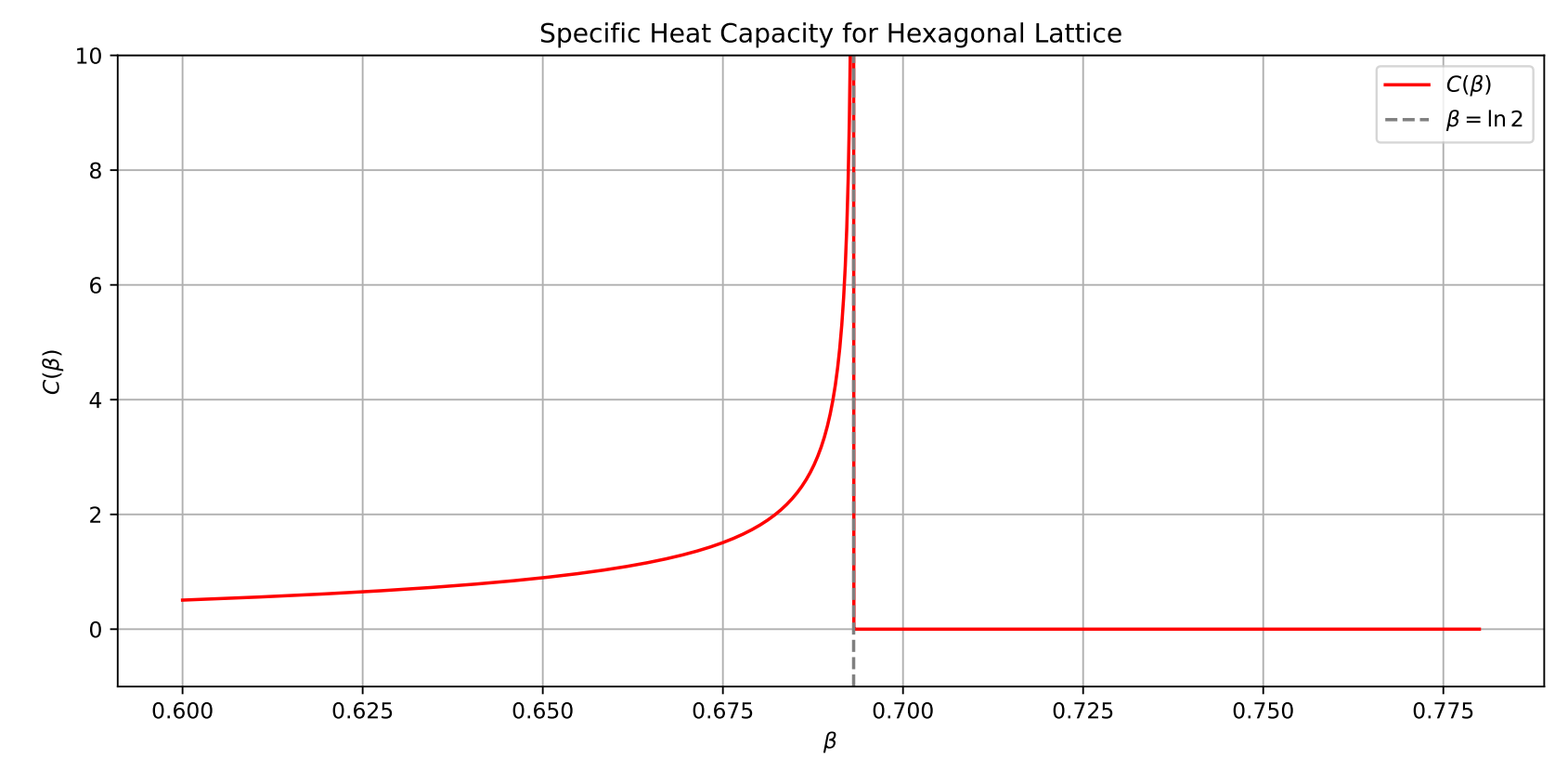
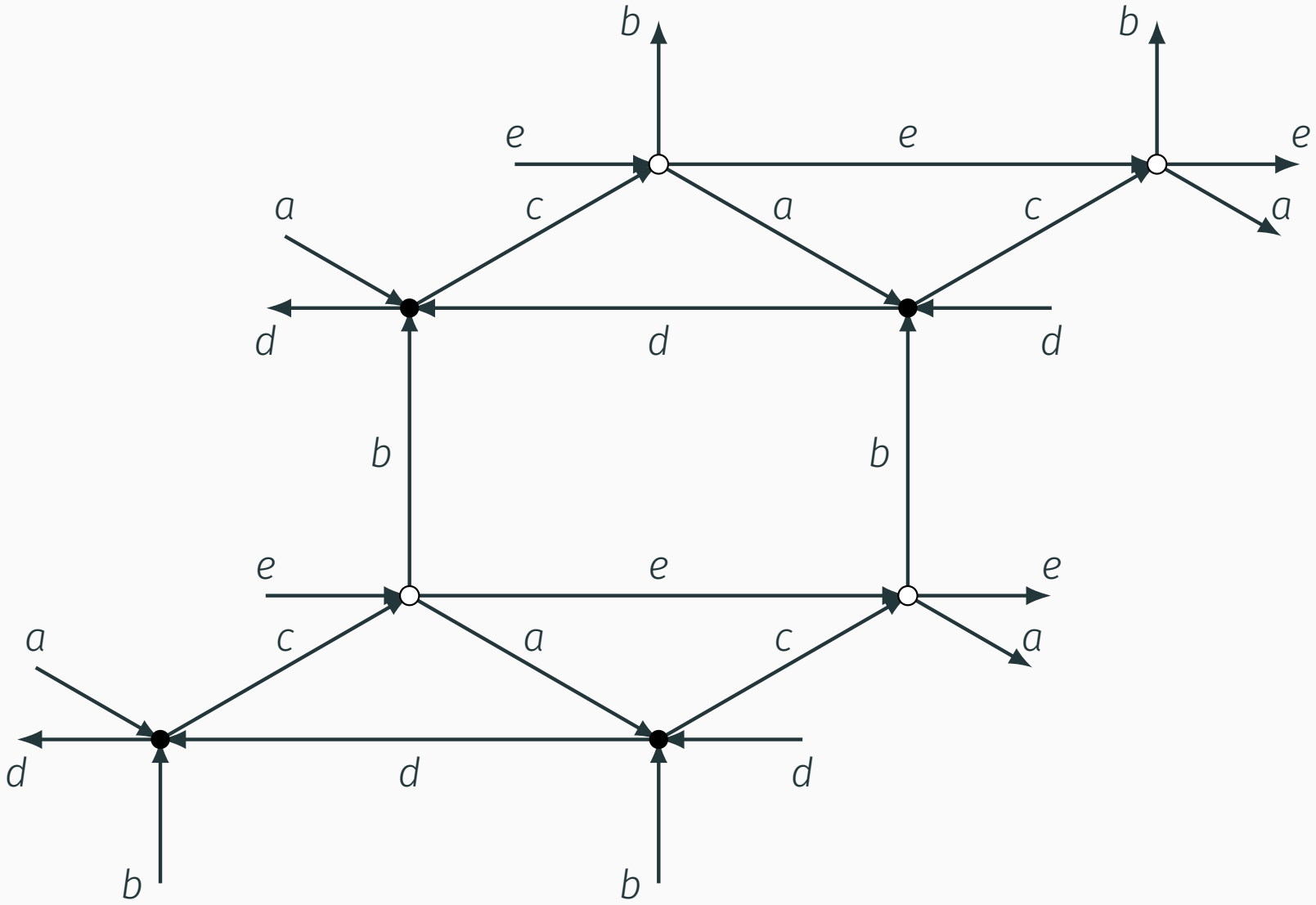


A Non-Bipartite Hexagonal Lattice

A Non-Bipartite Hexagonal Lattice: Setup



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- Applying the same method as in the bipartite case reduces the determinant of the Kasteleyn matrix to the following:

$$\det(K) = \prod_{n,m=1}^{N,M} \det \begin{pmatrix} -d\left(e^{2\pi i n/N} - e^{-2\pi i n/N}\right) & c - ae^{-2\pi i n/N} - be^{2\pi i m/M} \\ -\left(c - ae^{2\pi i n/N} - be^{-2\pi i m/M}\right) & \mathrm{e}\left(e^{2\pi i n/N} - e^{-2\pi i n/N}\right) \end{pmatrix} \tag{22}$$

$$= \prod_{n,m=1}^{N,M} \left(4\mathrm{e}d \sin^2\left(\frac{2\pi n}{N}\right) + \left|p^{\mathrm{Hex}}\left(e^{2\pi i n/N}, e^{2\pi i m/M}\right)\right|^2\right) \tag{23}$$

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- The full integral for W in the continuum limit looks like the following:

$$W = \frac{1}{2} \iint \frac{d\theta d\phi}{4\pi^2} \ln\left(4ed \sin^2\theta + c^2 + a^2 + b^2 - 2c(a\cos\theta + b\cos\phi) + 2ab\cos(\theta - \phi)\right) \tag{24}$$

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$$= \prod_{n,m=1}^{N,M} \left(4\mathbf{e}d\sin^2\left(\frac{2\pi n}{N}\right) + \left|p^{\mathrm{Hex}}\left(e^{2\pi i n/N}, e^{2\pi i m/M}\right)\right|^2\right) \tag{23}$$

- The full integral for W in the continuum limit looks like the following:

$$W = \frac{1}{2}\iint \frac{d\theta d\phi}{4\pi^2} \ln\left(4ed\sin^2\theta + c^2 + a^2 + b^2 - 2c(a\cos\theta + b\cos\phi) + 2ab\cos(\theta - \phi)\right) \tag{24}$$

- W was evaluated numerically to compute thermodynamic quantities of the lattice for $c = d = e = 1$ and $a = b = e^{-\beta}$

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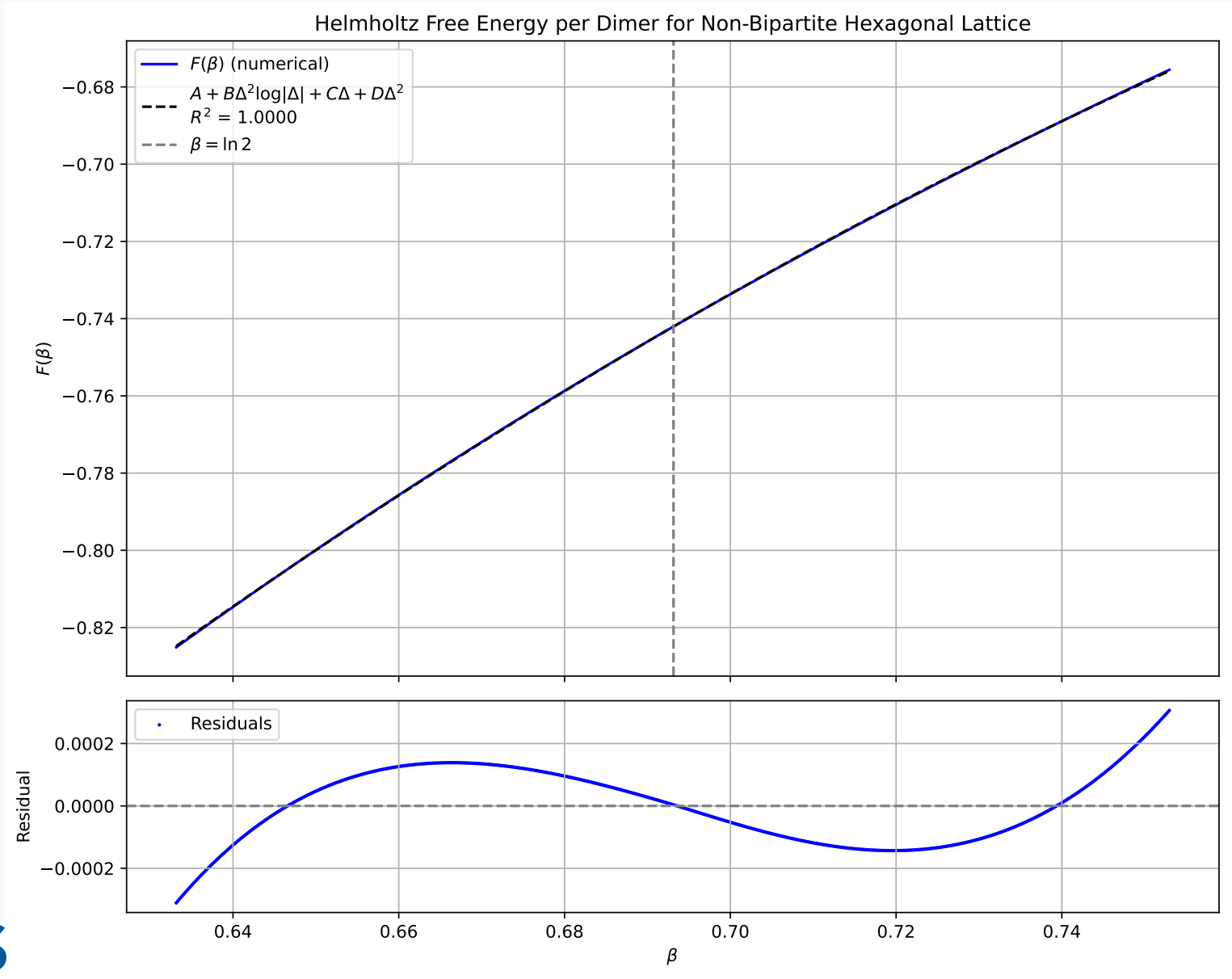
A Non-Bipartite Hexagonal Lattice: Thermodynamic Quantities



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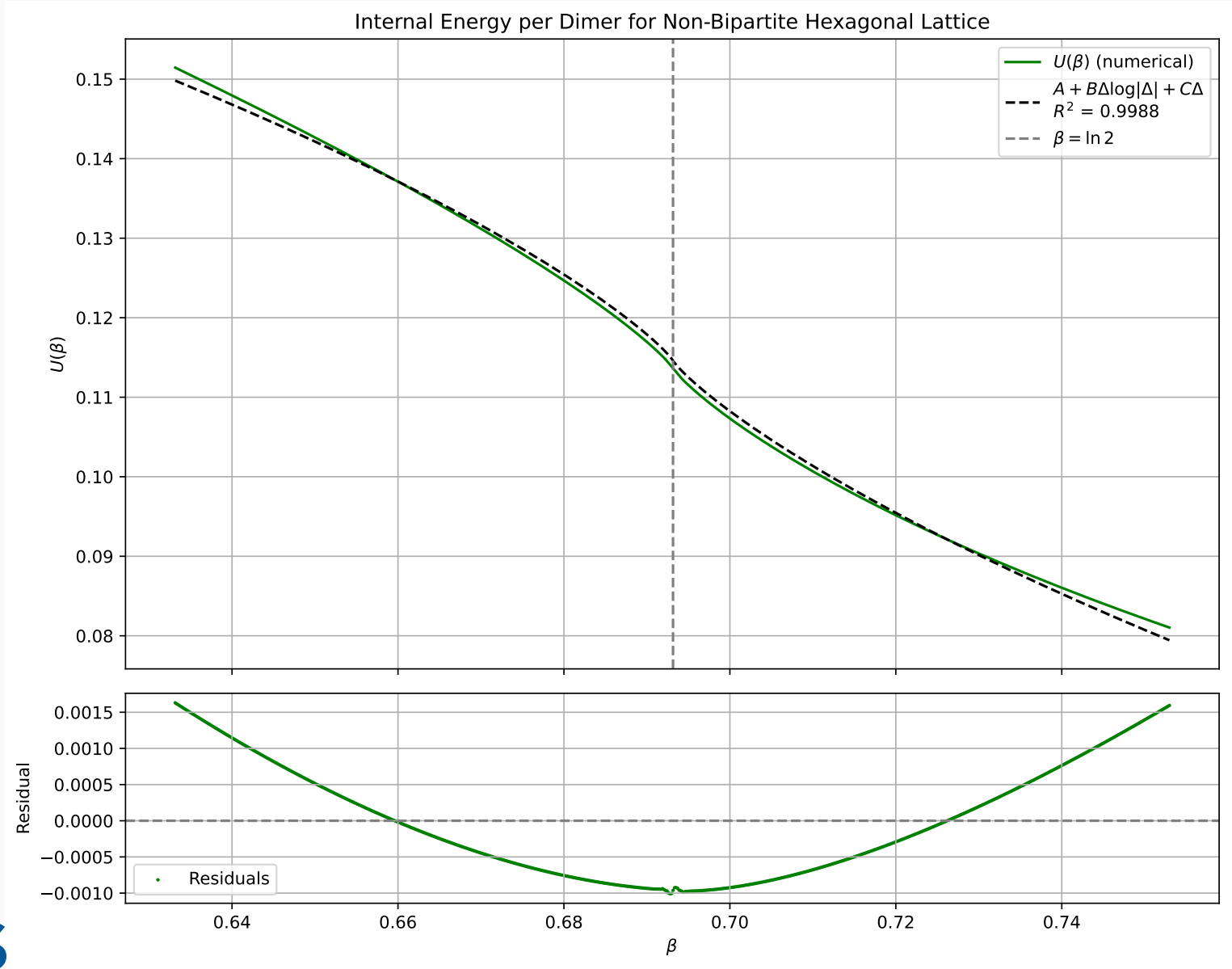

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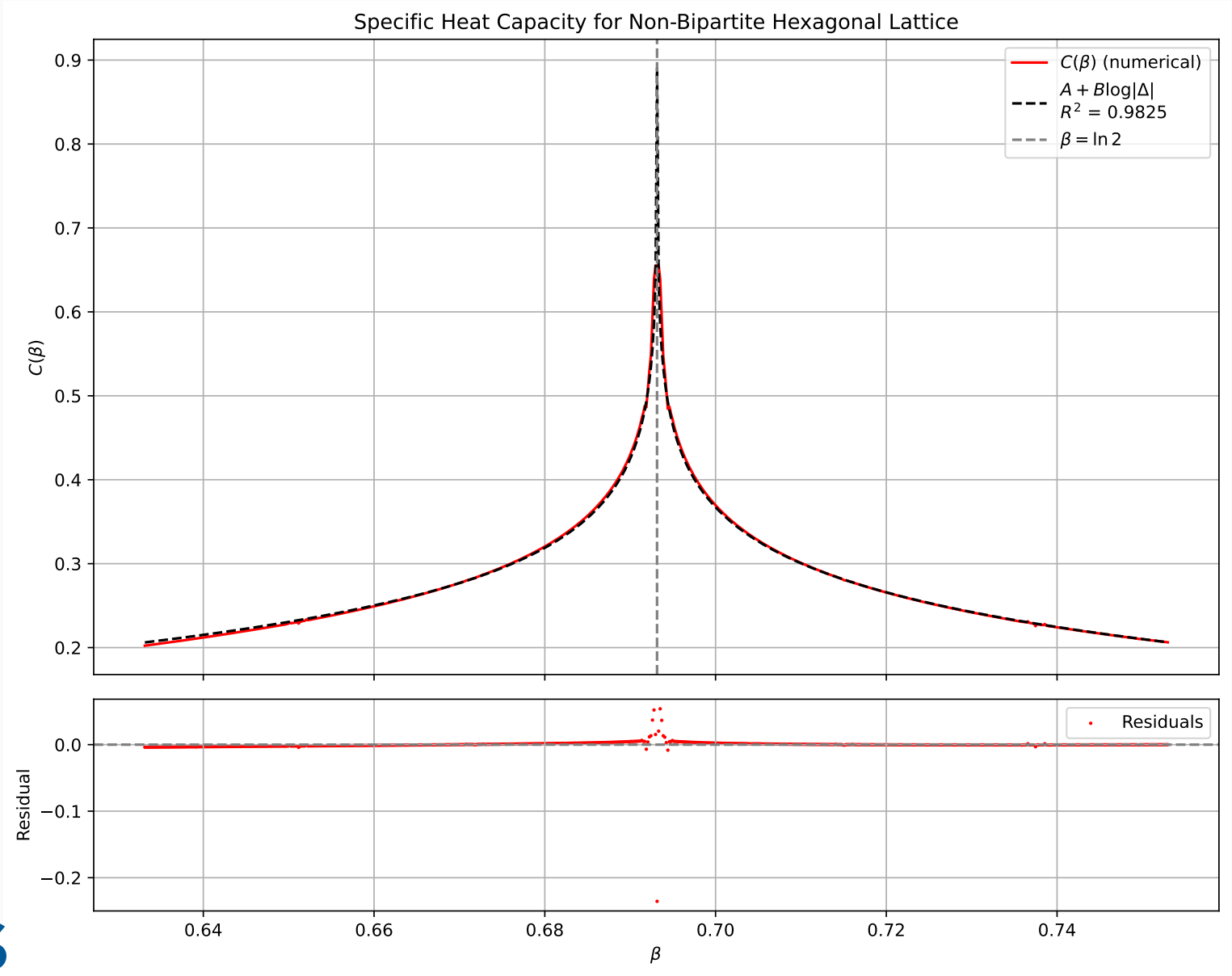

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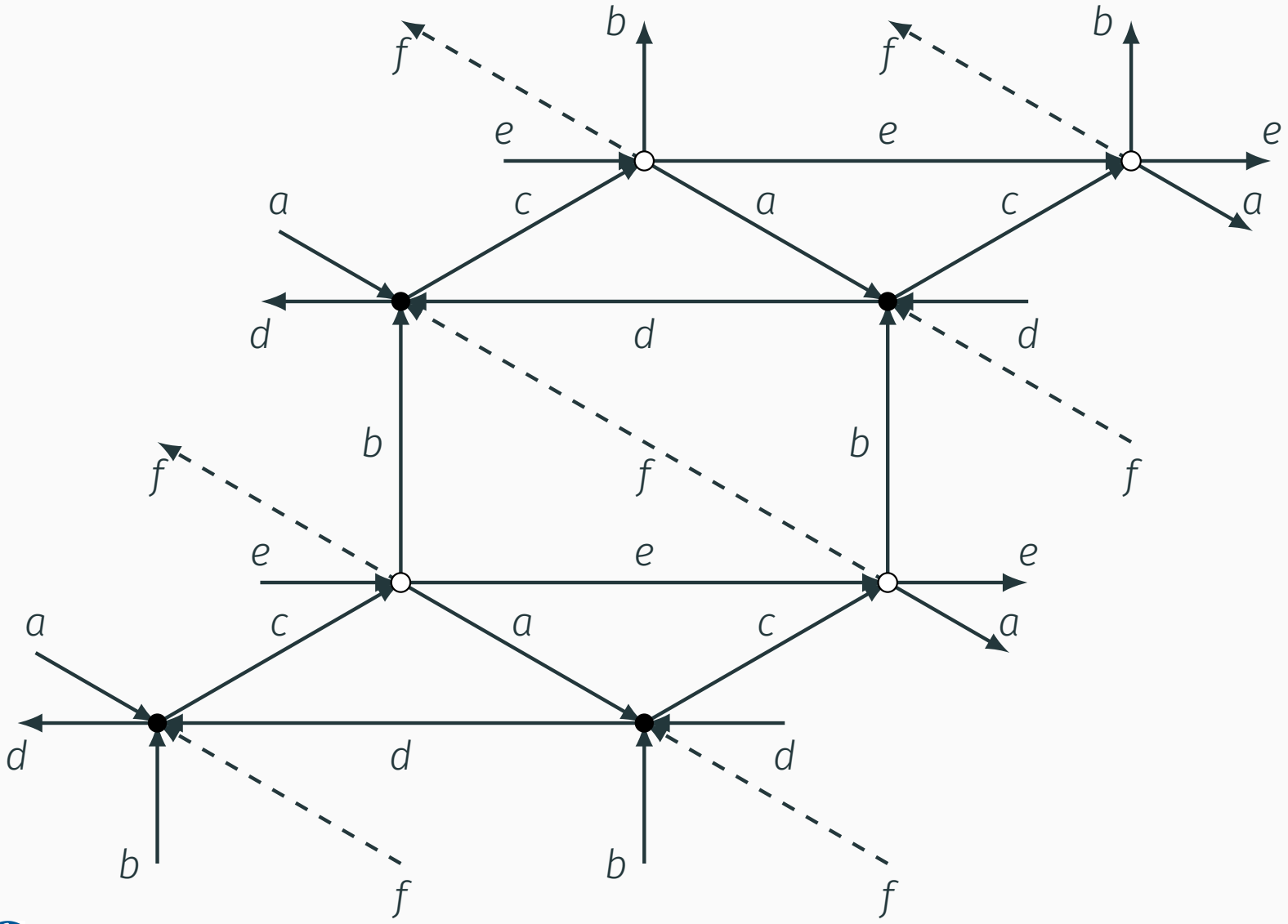
- This implies that the dimer model on this non-bipartite hexagonal lattice is in the same universality class as the Ising model [5].

The Triangular Lattice

Triangular Lattice: Setup



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Triangular Lattice: Setup

- Using the same machinery as before, the determinant of the Kasteleyn matrix to the following:, where again $\theta = \frac{2\pi n}{N}$ and $\phi = \frac{2\pi m}{M}$

$$\det(K) = \prod_{n,m=1}^{N,M} \det \begin{pmatrix} -d\left(e^{i\theta} - e^{-i\theta}\right) & p^{\text{Hex}}\left(e^{i\theta}, e^{i\phi}\right) - fe^{i(\theta+\phi)} \\ -\left(p^{\text{Hex}}\left(e^{-i\theta}, e^{-\phi}\right) - fe^{-i(\theta+\phi)}\right) & \mathbf{e}\left(e^{i\theta} - e^{-i\theta}\right) \end{pmatrix} \tag{25}$$

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$$W = \frac{1}{2}\oiint \frac{d\theta d\phi}{4\pi^2} \ln\left(4ed\sin^2\theta + c^2 + a^2 + b^2 + f^2 - 2c(a\cos\theta + b\cos\phi + \right. \tag{27}$$

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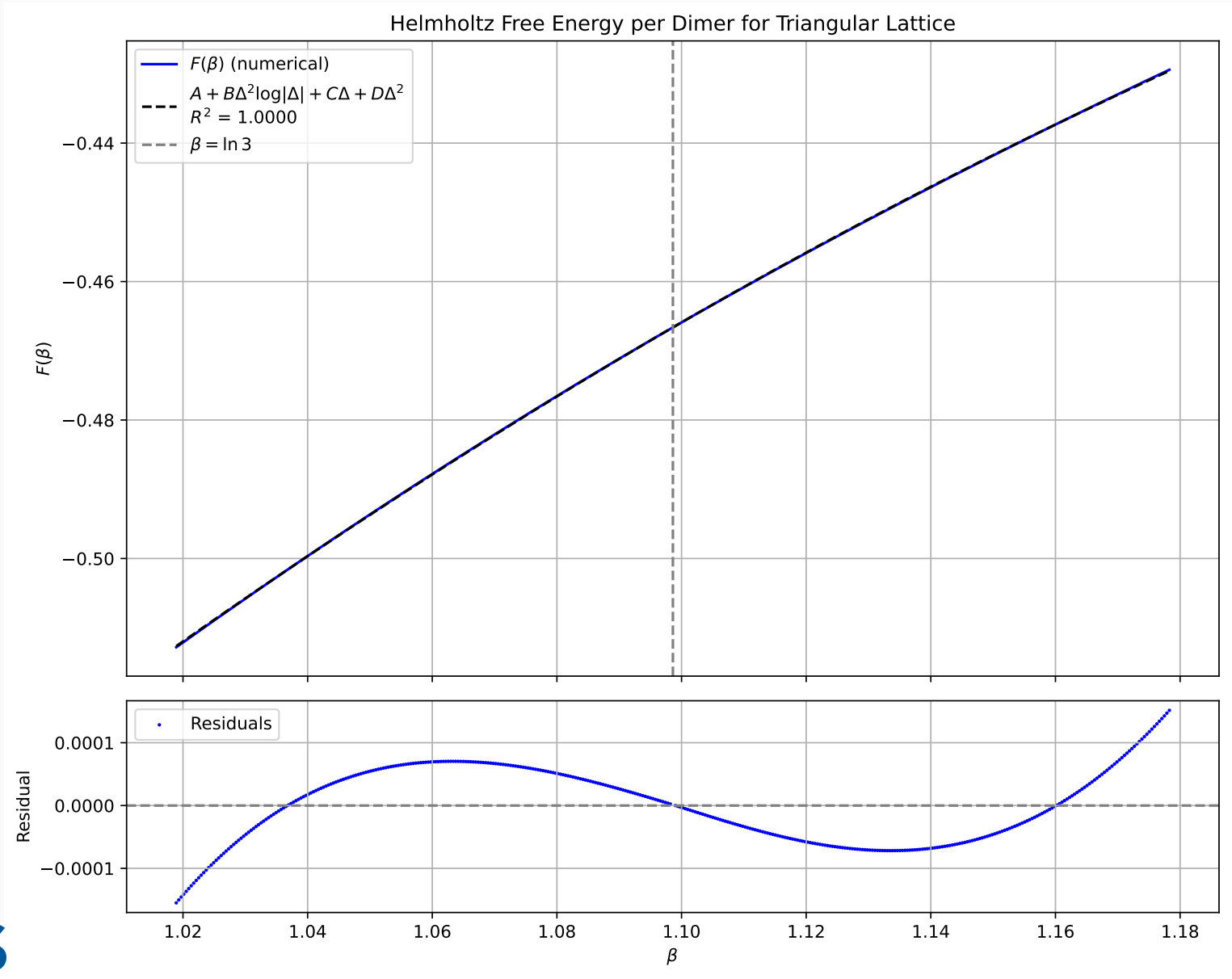

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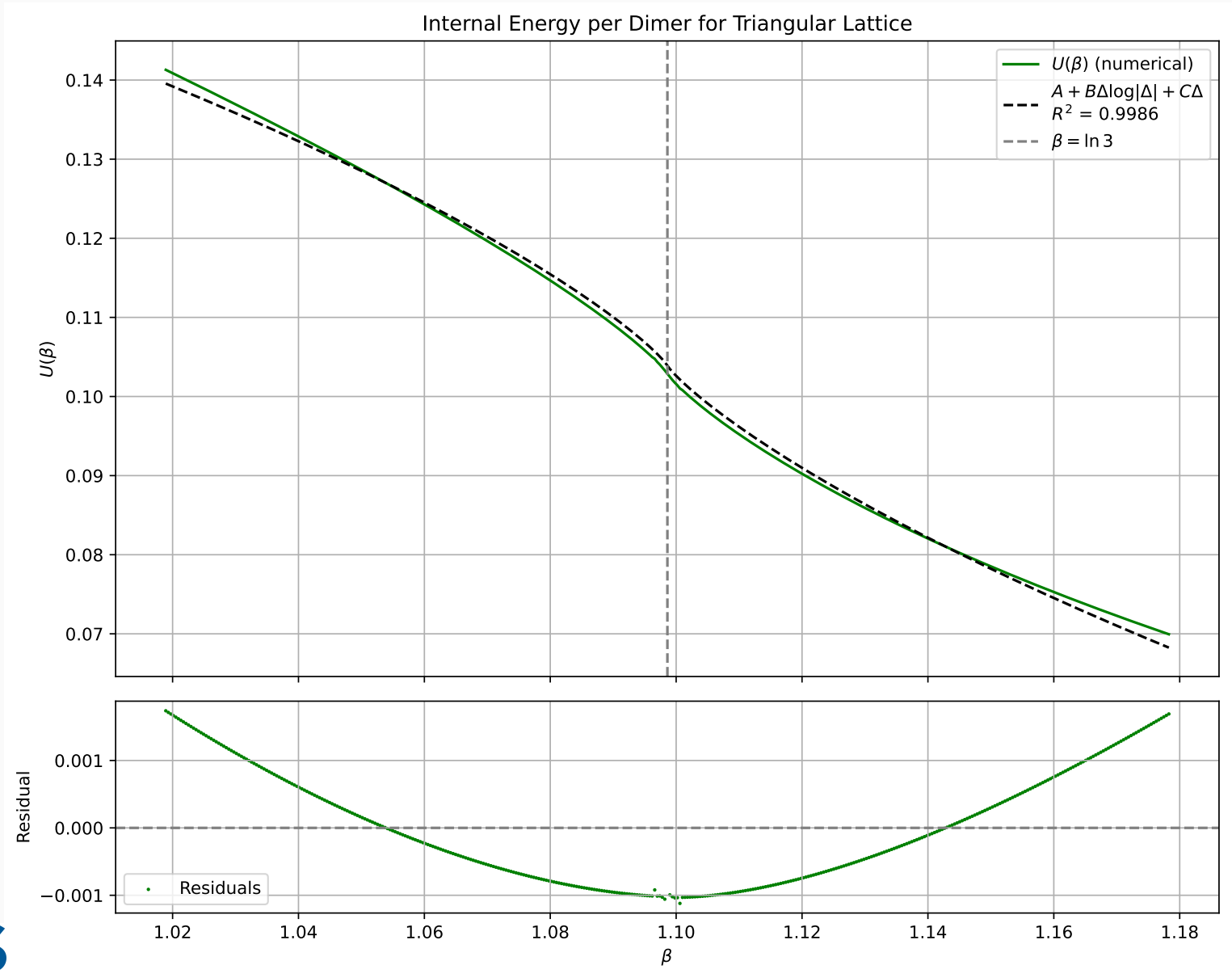

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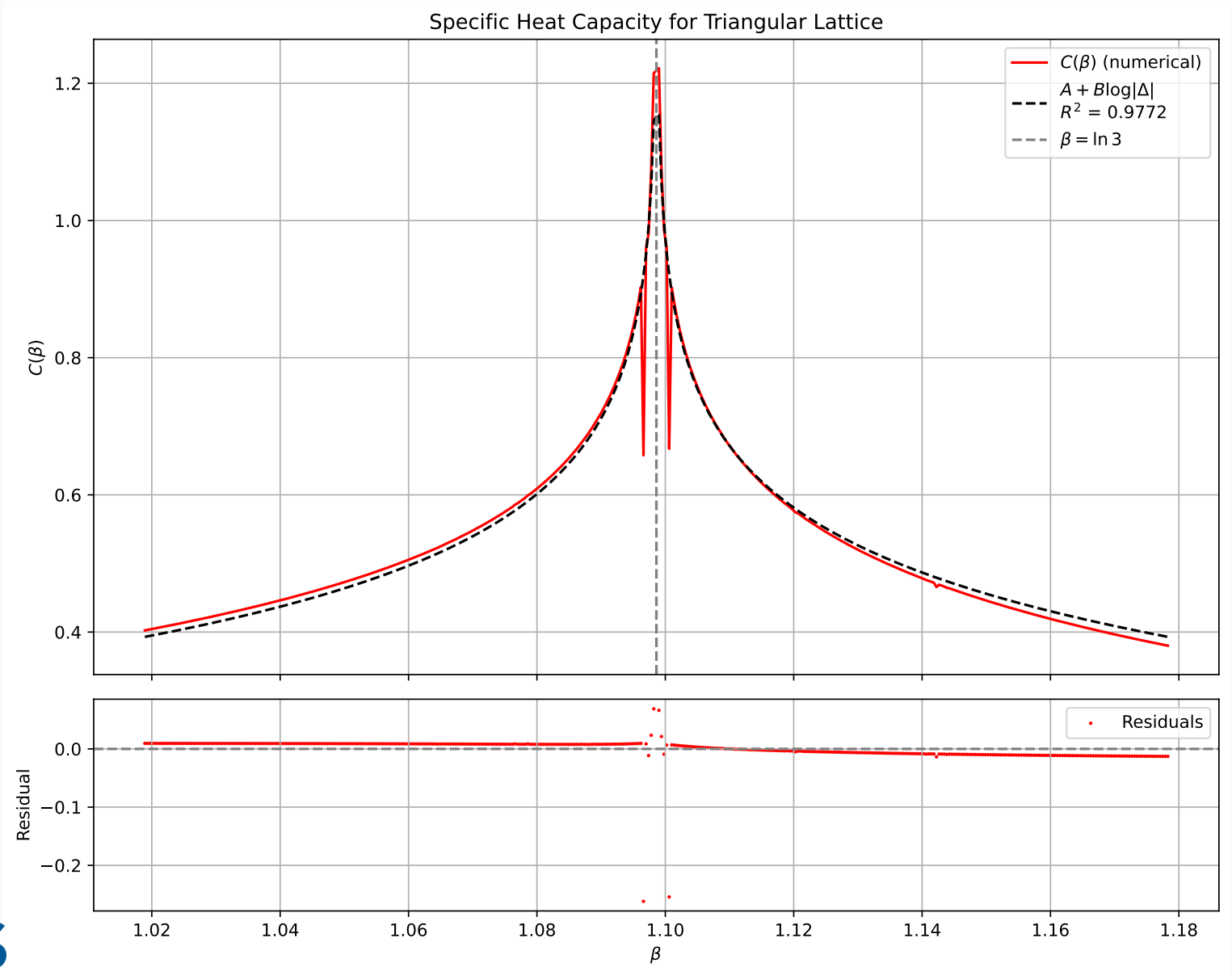

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A Bipartite Hexagonal Lattice with a Defect

Lattice with a Defect: Setup

- Now we wish to return to the original bipartite hexagonal lattice, but instead of having every bond of the same type having the same activity, we wish to study the effects of introducing a defect or rather a line of defects into the lattice.

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- Let's again attempt to visualise this one the 2×2 lattice example.

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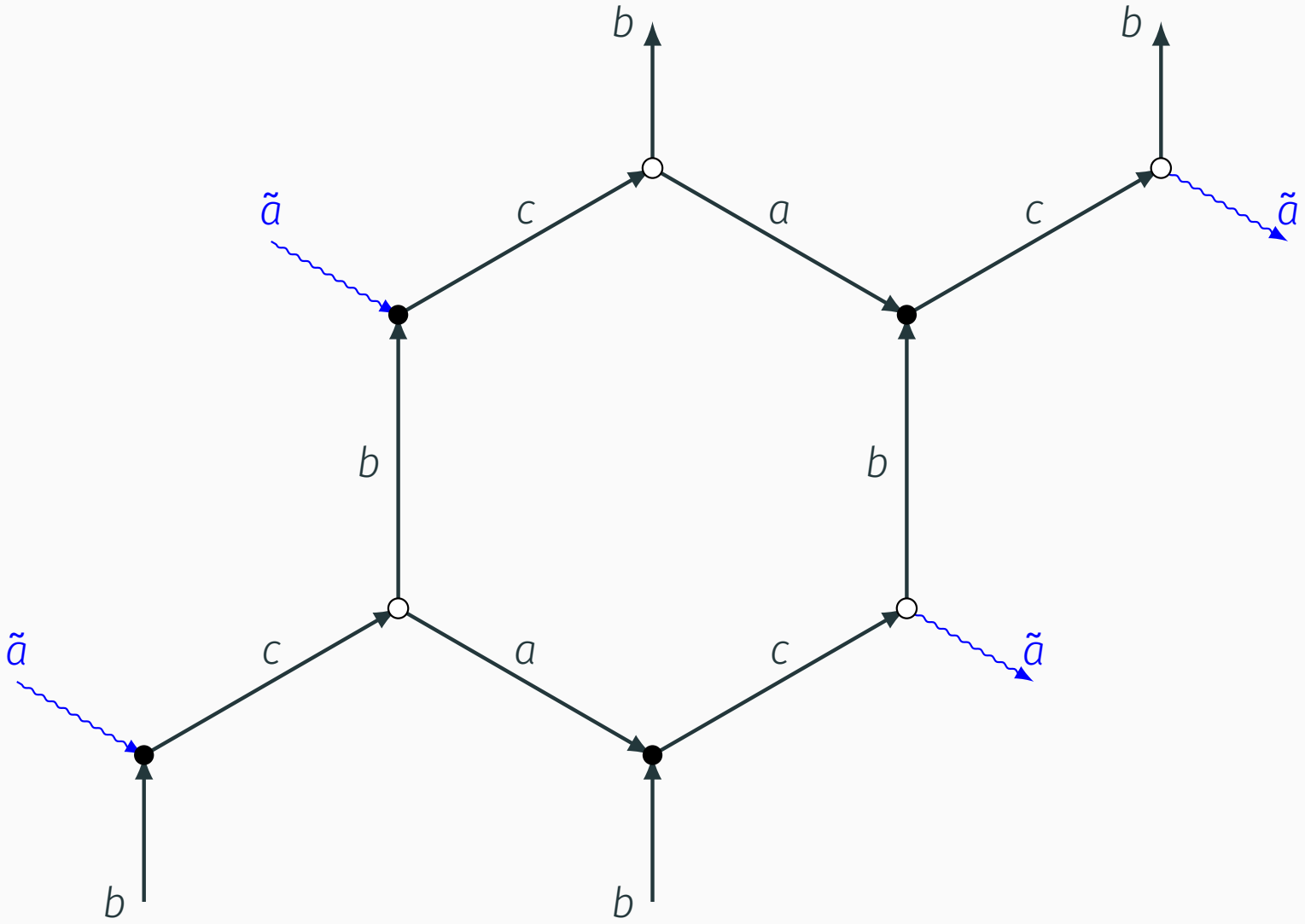

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Lattice with a Defect: The Partition Function

- Now, the partition function takes the following form:

$$\det(A_{u,v}) = \prod_{m=1}^{M} (c - be^{2\pi i(m+v)/M})^N - \tilde{a}a^{N-1}e^{2\pi iu} \tag{29}$$

$$\mathcal{Z} = \frac{1}{2} \sum_{u,v=0}^{1/2} e^{2\pi i(1/2+u+v+2uv)} \det(A_{u,v}) \tag{30}$$

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- Some values of $\tilde{a}$ have non-trivial consequences for the partition function, such as being zero- or negative-valued.

Conclusion

Summary

- Dimer models are interesting as an exactly solvable model that can teach us about phase transitions and other phenomena in physics and mathematics.

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Summary

- Dimer models are interesting as an exactly solvable model that can teach us about phase transitions and other phenomena in physics and mathematics.
- Dimer models on non-bipartite lattices tend to show Ising-like behaviour, exhibiting a continuous or second-order phase transition at the critical point.
- Defects on different lattices can lead to non-trivial consequences for the partition function and therefore the thermodynamic properties of those lattices.

Acknowledgements

I am very grateful to the Dublin Institute for Advanced Studies for the opportunity to do this project and the support I was given throughout it. In particular, I would like to thank my supervisor Prof. Denjoe O'Connor for his guidance during the project, as well as the programme coordinators Dr. Dmitry Noshchenko and Dr. Sabine Harribey.

Questions?

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